

DIAGRAMMATICS FOR SOERHEL CATEGORIES

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CONTENTS

1. Introduction	2
2. Preliminaries	4
2.1. Hecke algebra	4
2.2. Soergel bimodules	6
2.3. Categories of Soergel bimodules	8
2.4. Categorification dictionary	10
2.5. Diagrammatic calculus for bimodule maps	11
2.6. Methodology	12
3. Definition of \mathcal{DC}	14
3.1. The category \mathcal{DC}_1 : one color	14
3.2. The category \mathcal{DC}_1 : adjacent colors	22
3.3. The category \mathcal{DC}_1 : distant colors	25
3.4. Graphical terminology	27
4. Consequences	27
4.1. One color reductions	29
4.2. i -colored moves	31
4.3. \mathcal{F}_1 is fully faithful	33
4.4. Color elimination	37
4.5. The e_1 quotient	38
5. Proofs	39
5.1. \mathcal{F}_1 is a functor	39
5.2. Graphical proofs	44
References	52

ABSTRACT. The monoidal category of Soergel bimodules can be thought of as a categorification of the Hecke algebra of a finite Weyl group. We present this category, when the Weyl group is the symmetric group, in the language of planar diagrams with local generators and local defining relations.

1. INTRODUCTION

In the paper [39] Soergel gave a combinatorial description of a certain category of Harish-Chandra bimodules over a simple Lie algebra \mathfrak{g} . This category was and continues to be of primary interest in the infinite-dimensional representation theory of simple Lie algebras. Soergel described the category anew as a full subcategory of bimodules over a certain ring R . The objects in this subcategory are now commonly called Soergel bimodules. The category of Soergel bimodules is additive and monoidal, and Soergel constructed an isomorphism between the Grothendieck ring of his category and the integral form of the Hecke algebra of the Weyl group \mathcal{W} of \mathfrak{g} . Hence, Soergel's construction provides a categorification of the Hecke algebra.

Given a k -dimensional \mathbb{C} -vector space V and a generic $q \in \mathbb{C}$, there are commuting actions of the quantum group $U_q(\mathfrak{sl}_k)$ and the Hecke algebra \mathcal{H} of the symmetric group S_n on $V^{\otimes n}$. These actions turn the quotient $U_q(\mathfrak{sl}_k)/J_1$ of the quantum group and \mathcal{H}/J_2 of the Hecke algebra by the kernels of these actions into a dual pair. A categorical realization of the triple $(V^{\otimes n}, U_q(\mathfrak{sl}_k)/J_1, \mathcal{H}/J_2)$ was given by Grojnowski and Lusztig [16] via categories of perverse sheaves on products of flag and partial flag varieties, also see [5, 15, 6, 14].

Many foundational ideas about categorification were put forward by Igor Frenkel in the early 90's (a small fraction of these ideas formed a part of the paper [10]). In particular, Frenkel conjectured [13] that quantum groups and not just their finite-dimensional quotients $U_q(\mathfrak{sl}_k)/J_1$ can be categorified. These conjectures remained open until recently, when categorifications of quantum \mathfrak{sl}_2 and \mathfrak{sl}_k were discovered in [26] and [22], with a related but different approach developed in [8, 38]. In the categorifications [26, 22] of quantum groups, 2-morphisms are given by linear combinations of planar diagrams, modulo local relations.

The parallel objective would be to categorify the Hecke algebra in the same spirit, using planar diagrams. Soergel had already provided a categorification, so it remains to ask whether his category can be rephrased diagrammatically. Diagrammatics should also provide a presentation of the category by generators and relations. A similar question was recently posed by Libedinsky [29], who essentially produced such a description for categorifications of Hecke algebras associated to "right-angled" Coxeter systems.

Here we answer this question positively in the case of the Hecke algebra associated to the symmetric group. This is, of course, the Hecke algebra that appears in the Schur-Weyl duality for $V^{\otimes n}$. For notational convenience, we use $n + 1$, not n , as our parameter and define a diagrammatical version \mathcal{DC} of the category of Soergel bimodules that categorifies the Hecke algebra of the symmetric group S_{n+1} .

In some sense, diagrammatic categorifications are very "low-tech," in that they can be described easily and do not rely on heavy machinery. While the proof that Soergel bimodules categorify the Hecke algebra is based on elaborate commutative algebra, showing that indecomposable bimodules descend to the Kazhdan-Lusztig basis of the Hecke algebra utilizes Kazhdan-Lusztig theory [18, 19]. This, in turn, is intrinsically linked to fundamental developments in representation theory that relate highest weight

categories with D-modules on flag manifolds [3, 7], as well as the theory of perverse sheaves [4] and etale cohomology. One hopes that a diagrammatic approach will help to visualize and work with these sophisticated constructions.

We start with an intermediate category \mathcal{DC}_1 whose objects are finite sequences $\underline{i} = i_1 \dots i_d$ of numbers between 1 and n . An object is represented graphically by marking d points in the standard position (say, having coordinates $1, \dots, d$) on the x -axis and assigning labels i_1, \dots, i_d to marked points from left to right. Morphisms between \underline{i} and \underline{j} are given by linear combinations (with coefficients in a ground field \mathbb{k}) of planar diagrams embedded in the strip $\mathbb{R} \times [0, 1]$. These diagrams are decorated planar graphs, where edges may extend to the boundary $\mathbb{R} \times \{0, 1\}$. Each edge carries a label between 1 and n , so that the induced labellings of the lower and upper boundaries are \underline{i} and \underline{j} , respectively. In the interior of the strip, we allow

- vertices of valence 1,
- vertices of valence 3 with all 3 edges carrying the same label,
- vertices of valence 4 seen as intersections of i and j -labelled lines with $|i - j| > 1$,
- vertices of valence 6 with the edge labelling $i, i + 1, i, i + 1, i, i + 1$, reading clockwise around the vertex.

Furthermore, boxes labelled by numbers between 1 and $n + 1$ are allowed to float in the regions of the graph. We impose a set of local relations on linear combinations of these diagrams, including invariance of diagrams under all isotopies. A subset of the relations says that i is a Frobenius object in the category \mathcal{DC}_1 . These diagrammatic generators and relations were guessed by studying generating morphisms between tensor products of basic Soergel bimodules B_i and writing down relations between these morphisms.

The space of morphisms in \mathcal{DC}_1 between \underline{i} and \underline{j} is naturally a graded vector space. Allowing grading shifts and direct sums of objects, then restricting to grading-preserving morphisms, and finally passing to the Karoubian closure of the category results in a graded \mathbb{k} -linear additive monoidal category \mathcal{DC} . Our main result (Theorem 1 in Section 4) is an explicit equivalence between this category and the category \mathcal{SC} of Soergel bimodules.

The category \mathcal{DC} is monoidal, and can be viewed as a 2-category with a single object (where in a diagrammatic 2-category, regions are labelled by objects, and edges by 1-morphisms). This may make it easier to explore similarities with the categorifications of quantum groups in [22, 26], where regions of diagrams are labelled by integers in [26] and integral weights of \mathfrak{sl}_n in [22]. Boxes floating in regions are superficially analogous to floating bubbles of [22, 26]. Unlike the diagrammatic categorifications in [22, 26], our lines don't carry dots and are not oriented.

There is another way to view our diagrammatics, which is not developed in this paper. Rouquier [36, 37] defined an action of the Coxeter braid group associated to \mathcal{W} on the category of complexes of Soergel bimodules up to homotopies. This action was later used in an alternative construction [21] of a triply-graded link homology theory [23] categorifying the HOMFLY-PT polynomial [11, 23, 35, 43]. In this approach, a product Soergel bimodule $B_{i_1} \otimes \dots \otimes B_{i_d}$ is depicted by a planar diagram given by concatenating

elementary planar diagrams lying in the xy -plane that consist of $n + 1$ strands going up, with i and $i + 1$ -st strands merging and splitting, see [21, Figures 2,3]. Morphisms between product bimodules can be realized by linear combinations of foams – decorated two-dimensional CW-complexes embedded in \mathbb{R}^3 with suitable boundary conditions. Foams have been implicit throughout papers on triply-graded link homology (see [31] for instance, where various arrows between planar diagrams can be implemented by foams), and explicitly appear in the papers on their doubly-graded cousins, see [30, 33] and references therein.

Foams are 3-dimensional objects – they are two-dimensional CW-complexes embedded in \mathbb{R}^3 that produce cobordisms along the z -axis direction between planar objects corresponding to product Soergel bimodules. The planar diagrams of our paper are two-dimensional encodings of these foams, essentially projections of the foams onto the yz -plane along the x -axis.

It was shown in [24] that the action of the braid group on the homotopy category of Soergel bimodules extends to a (projective) action of the category of braid cobordisms. Thus, the homotopy category of \mathcal{DC} produces invariants of braid cobordisms, so that our planar diagrammatics carry information about four-dimensional objects. This informational density indicates efficiency of such encodings.

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2. PRELIMINARIES

Henceforth we will fix a positive integer n . Indices i, j , and k will range over $1, \dots, n$ if not otherwise specified. Finite ordered sequences of such indices (allowing repetition) will be denoted $\underline{i} = i_1 \dots i_d$, as well as \underline{j} and \underline{k} . For sequences of length $d = 1$ where the single entry is i , we use i and \underline{i} interchangeably. Occasionally $i + 1$ will also be used as an index, and whenever this occurs we make the tacit assumption that $i \leq n - 1$ so that all indices used remain between 1 and n . The same goes for $i - 1$, $i + 2$ and the like. We denote the length 0 sequence by the empty set symbol \emptyset .

We work over a field \mathbb{k} , usually assuming that $\text{Char } \mathbb{k} \neq 2$, and sometimes specializing it to \mathbb{C} .

Given a noetherian ring R , the category $R\text{-molf-}R$ is the full subcategory of R -bimodules consisting of objects which are finitely generated as left R -modules. If R is graded, the category $R\text{-molf}_{\mathbb{Z}}\text{-}R$ is the analogous subcategory of graded R -bimodules and grading-preserving homomorphisms.

2.1. Hecke algebra. Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system of a finite Weyl group \mathcal{W} , with length function $l: \mathcal{W} \rightarrow \mathbb{N} = \{0, 1, 2, \dots\}$, and $e \in \mathcal{W}$ the identity. The Hecke algebra \mathcal{H} is an algebra over $\mathbb{Z}[t, t^{-1}]$ (we follow Soergel's use [39] of the variable t ; related variables are denoted in the literature by $v = t^{-1}$ and $q = t^2$), which is free as a module with basis T_w , $w \in \mathcal{W}$. Multiplication in this basis is given by $T_v T_w = T_{vw}$ when

$l(v) + l(w) = l(vw)$, and $T_s^2 = (t^2 - 1)T_s + t^2T_e$ for $s \in \mathcal{S}$. T_e is the identity element in \mathcal{H} and will often be written as 1.

In the case we are interested in presently, $\mathcal{W} = S_{n+1}$, and \mathcal{S} consists of the transpositions $s_i = (i, i+1)$ for $i = 1, \dots, n$. The element T_{s_i} will be denoted T_i . The Hecke algebra has a presentation over $\mathbb{Z}[t, t^{-1}]$, being generated by T_i subject to the relations

$$\begin{aligned} T_i^2 &= (t^2 - 1)T_i + t^2 \\ T_i T_j &= T_j T_i \text{ for } |i - j| \geq 2 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}. \end{aligned}$$

Clearly then, \mathcal{H} is also generated as an algebra by $b_i \stackrel{\text{def}}{=} C'_{s_i} = t^{-1}(T_i + 1)$, $1 \leq i \leq n$, and the relations above transform into

$$\begin{aligned} (2.1) \quad b_i^2 &= (t + t^{-1})b_i \\ (2.2) \quad b_i b_j &= b_j b_i \text{ for } |i - j| \geq 2 \\ (2.3) \quad b_i b_{i+1} b_i + b_{i+1} &= b_{i+1} b_i b_{i+1} + b_i. \end{aligned}$$

We often write the monomial $b_{i_1} b_{i_2} \cdots b_{i_d}$ as $b_{\mathbf{i}}$ where $\mathbf{i} = i_1 \dots i_d$. Notice that $b_\emptyset = 1$.

Let $a \mapsto \bar{a}$ be the involution of $\mathbb{Z}[t, t^{-1}]$ determined by $\bar{t} = t^{-1}$. It extends to an involution of \mathcal{H} given by

$$\overline{\sum a_w T_w} = \sum \bar{a}_w T_{w^{-1}}^{-1}$$

In particular, $\bar{T}_i = T_i^{-1} = t^{-2}T_i + t^{-2} - 1$.

Kazhdan and Lusztig [18] defined a pair of bases $\{C_w\}_{w \in \mathcal{W}}$ and $\{C'_w\}_{w \in \mathcal{W}}$ for \mathcal{H} , which immediately proved to be of fundamental importance for representation theory and combinatorics. The two bases are related via a suitable involution of \mathcal{H} , and the elements of the second Kazhdan-Lusztig basis $\{C'_w\}_w$ are determined by the two properties:

$$\begin{aligned} \overline{C'_w} &= C'_w, \\ C'_w &= t^{-l(w)} \sum_{y \leq w} P_{y,w} T_y, \end{aligned}$$

where $P_{y,w} \in \mathbb{Z}[t^2]$ has t -degree strictly less than $l(w) - l(y)$ for $y < w$ and $P_{w,w} = 1$. There is no simple formula expressing C'_w in terms of T_y , but observe that $C'_e = 1$ and $C'_{s_i} = b_i = t^{-1}(T_i + 1)$.

Let $\tau : \mathcal{H} \rightarrow \mathbb{Z}[t, t^{-1}]$ be the $\mathbb{Z}[t, t^{-1}]$ -linear map given by $\tau(T_e) = 1$ and $\tau(T_w) = 0$ if $w \neq e$. Thus, τ simply picks up the coefficient of T_e in x . The easily checked property $\tau(T_i x) = \tau(x T_i)$ for any $x \in \mathcal{H}$ implies that $\tau(xy) = \tau(yx)$, $\forall x, y \in \mathcal{H}$. The map τ turns \mathcal{H} into a symmetric Frobenius $\mathbb{Z}[t, t^{-1}]$ -algebra.

Denote by ω a t -antilinear antiinvolution $\omega : \mathcal{H} \rightarrow \mathcal{H}$ defined uniquely by $\omega(b_i) = b_i$. The antiinvolution and t -antilinearity conditions say that $\omega(xy) = \omega(y)\omega(x)$ and $\omega(ax) = \bar{a}\omega(x)$, for $x, y \in \mathcal{H}$ and $a \in \mathbb{Z}[t, t^{-1}]$.

Consider the pairing $(,): \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{Z}[t, t^{-1}]$ of \mathbb{Z} -modules given by

$$(x, y) = \overline{\tau(x\omega(y))}.$$

It satisfies the following properties:

- (1) The pairing is *semi-linear*, i.e. $(ax, y) = \overline{a}(x, y)$ while $(x, ay) = a(x, y)$, for $a \in \mathbb{Z}[t, t^{-1}]$.
- (2) b_i is self-adjoint, i.e. $(x, b_i y) = (b_i x, y)$ and $(x, y b_i) = (x b_i, y)$.
- (3) If $\underline{i} = i_1 \dots i_d$ with $i_1 < i_2 < \dots < i_d$ then $(1, b_{\underline{i}}) = t^d$. Such a monomial $b_{\underline{i}}$ is called an *increasing monomial*, and \underline{i} an *increasing sequence*. When $d = 0$, the sequence \underline{i} is empty and $(1, 1) = 1$.

Remark 2.1. It is not difficult to observe that $(,)$ is the unique form satisfying these three properties. This is because the Hecke algebra has a spanning set over $\mathbb{Z}[t, t^{-1}]$ consisting of monomials $b_{\underline{i}}$, and every monomial may be reduced, by cycling the last b_i to the beginning and by applying the Hecke algebra relations, to an increasing monomial. This is a simple combinatorial argument that we leave to the reader.

Let $\varepsilon: \mathcal{H} \rightarrow \mathbb{Z}[t, t^{-1}]$ be given by $\varepsilon(y) = \overline{\tau(\omega(y))}$. This is a $\mathbb{Z}[t, t^{-1}]$ -linear map, equipping \mathcal{H} with a symmetric Frobenius $\mathbb{Z}[t, t^{-1}]$ -algebra structure, just like τ did. We have $\varepsilon(y) = (1, y)$ and $(x, y) = \varepsilon(\omega(x)y)$.

2.2. Soergel bimodules. In [39] Soergel introduced a category of bimodules which categorified the Hecke algebra, and later generalized his construction to any Coxeter group \mathcal{W} [42]. Within the category $R\text{-molf}_{\mathbb{Z}}\text{-}R$, for R a certain graded \mathbb{k} -algebra (\mathbb{k} an infinite field of characteristic $\neq 2$), he identified indecomposable modules B_w for $w \in \mathcal{W}$, such that the only indecomposable summands of tensor products of B_w 's are $B_{w'}$ for $w' \in \mathcal{W}$. Thus, the subcategory of $R\text{-molf}_{\mathbb{Z}}\text{-}R$ generated additively by the B_w has a tensor product, and its Grothendieck ring is isomorphic to \mathcal{H} , under the isomorphism sending C'_s to $[B_s]$. Moreover, every B_w shows up as a summand of some tensor product of various B_s for $s \in \mathcal{S}$. While the general B_w may be difficult to describe, B_s has an easy description.

It is conjectured in [42] that this isomorphism sends $[B_w]$ to C'_w for all $w \in \mathcal{W}$, and it is proven for $\mathbb{k} = \mathbb{C}$ and \mathcal{W} a Weyl group in [39].

Henceforth we specialize to the case where $\mathcal{W} = S_{n+1}$ and $\mathcal{S} = \{s_i\}$. Here one may also define the Soergel bimodule B'_w instead of B_w , as discussed in [21]. The B_w are obtained by taking the fundamental n -dimensional representation V of S_{n+1} and looking at the coordinate rings of twisted diagonals in $V \times V$. The B'_w are defined analogously with the standard $(n+1)$ -dimensional representation, which is V plus the trivial representation. Effectively, $B'_w \cong B_w \otimes \mathbb{k}[y]$ and the entire story of B_w translates to B'_w by base extension. Both versions will categorify the Hecke algebra, and we will interest ourselves with the B'_w story below; to deal with B_w instead, see Section 4.5. Since we only use B'_w below, we will denote them as B_w instead to avoid an apostrophe catastrophe.

With these conventions, we now make the story explicit. Let $R = \mathbb{k}[x_1, x_2, \dots, x_{n+1}]$ be the ring of polynomials in $n+1$ variables, with the natural action of S_{n+1} . The ring R is graded, with $\deg(x_i) = 2$. If W is the subgroup of S_{n+1} generated by transpositions $\{s_{i_1}, \dots, s_{i_r}\}$, then we denote the ring of invariants under W as R^{i_1, \dots, i_r} or R^W . Thus R^i are the invariants under the transposition $(i, i+1)$.

Since R is an R^W -algebra, \otimes_{R^W} is a bifunctor sending two R -bimodules to an R -bimodule. Henceforth \otimes with no subscript denotes tensoring over R , while $\otimes_{i_1, \dots, i_r}$ denotes tensoring over the subring R^{i_1, \dots, i_r} . Most commonly we will just use \otimes_i for various indices i .

The Soergel bimodule B_i is $R \otimes_i R\{-1\}$, where $\{m\}$ denotes a grading shift. We denote by $B_{\underline{i}}$ the tensor product $B_{i_1} \otimes B_{i_2} \otimes \dots \otimes B_{i_d}$. Note that $B_{\emptyset} = R$ and

$$\begin{aligned} B_{\underline{i}} &= (R \otimes_{i_1} R\{-1\}) \otimes (R \otimes_{i_2} R\{-1\}) \otimes \dots \\ &= R \otimes_{i_1} R \otimes_{i_2} \dots \otimes_{i_d} R\{-d\}. \end{aligned}$$

An element of this module can be written as a sum of terms given by $d+1$ polynomials, one in each slot separated by the tensors. Multiplication by an element of R in any particular slot is an R -bimodule endomorphism.

For each i there is a map of graded vector spaces $\partial_i: R \rightarrow R^i$ sending $f \mapsto \frac{f - s_i(f)}{x_i - x_{i+1}}$ which lowers the degree by two. Since $\partial_i(f)$ is s_i -invariant, it is not hard to see that $P_i(f) = f - x_i \partial_i(f)$ is also s_i -invariant. Since $f = P_i(f) + x_i \partial_i(f)$, this implies that R is a free graded R^i -module of rank two, with homogeneous generators 1 and x_i .

Likewise, B_i is a free graded R -module with a basis $\{1 \otimes 1, 1 \otimes x_i\}$, under the action of R by left multiplication. A product element decomposes in this basis as

$$f \otimes g = f \otimes P_i(g) + f \otimes x_i \partial_i(g) = P_i(g) f \otimes 1 + \partial_i(g) f \otimes x_i.$$

In particular, this gives an isomorphism $B_i \cong R\{-1\} \oplus R\{1\}$ as graded left R -modules. Iterating and ignoring the grading, we see that $B_{\underline{i}}$ is a free left R -module of rank 2^d , when \underline{i} has length d . Rewriting a term $1 \otimes g$ as a sum of terms like above will happen often, and we refer to it as “sliding” the polynomial g across the tensor. If g is s_i -invariant then it may be slid across leaving nothing behind, while an arbitrary g leaves terms with at either 1 or x_i behind (alternatively, we may choose to leave 1 and x_{i+1} behind, if it is more convenient).

We also remark on spanning sets for $B_{\underline{i}}$. For instance, $B_i \otimes B_i$ has a spanning set $\{f \otimes 1 \otimes g, f \otimes x_i \otimes g\}$, where f, g range over all monomials in R . The bimodule $B_i \otimes B_j$ for $j \neq i$ has a spanning set $\{f \otimes 1 \otimes g\}$, over all monomials f, g , since any polynomial in the middle can be slid to the left leaving at most x_i behind (or x_{i+1} , which we choose when $j = i-1$), and that can be slid to the right; thus $1 \otimes 1 \otimes 1$ generates it as a R -bimodule. The bimodule $B_i \otimes B_j \otimes B_k \otimes B_i$ has a spanning set $\{f \otimes 1 \otimes 1 \otimes 1 \otimes g, f \otimes x_i \otimes 1 \otimes 1 \otimes g\}$, over all monomials f, g . This is because all polynomials anywhere between the two i tensors may be slid out, leaving x_i somewhere in-between. As an exercise, the reader may generalize this argument to an arbitrary $B_{\underline{i}}$ and find a spanning set as a R -bimodule, consisting of $2^{m(\underline{i})}$ terms, where $m(\underline{i})$ is

the number of pairs $1 \leq r < s \leq d$ such that $i_r = i_s$ and $i_t \neq i_s$ for t between r and s , which is equal to d minus the number of distinct indices in \underline{i} .

For more information on Soergel bimodules and their applications we refer the reader to [39, 40, 41, 12, 21, 27, 28, 36, 44, 46] and references therein.

2.3. Categories of Soergel bimodules. Several subcategories of $R\text{-molf-}R$ and $R\text{-molf}_{\mathbb{Z}}\text{-}R$ will play a role in what follows. Let \mathcal{SC}_1 be the full subcategory of $R\text{-molf-}R$ whose objects consist of $B_{\underline{i}}$ for all sequences of indices \underline{i} . Since R is a commutative ring, the Hom spaces in \mathcal{SC}_1 are in fact enriched in $R\text{-molf}_{\mathbb{Z}}\text{-}R$. Let \mathcal{SC}_2 be the subcategory of $R\text{-molf}_{\mathbb{Z}}\text{-}R$ whose objects are finite direct sums of various graded shifts of objects in \mathcal{SC}_1 and the morphisms are all grading-preserving bimodule homomorphisms. Finally, let \mathcal{SC} be the Karoubi envelope of \mathcal{SC}_2 , a category equivalent to the full subcategory of $R\text{-molf}_{\mathbb{Z}}\text{-}R$ which contains all summands of objects of \mathcal{SC}_2 :

$$\mathcal{SC}_1 \xrightarrow{\text{Grading shifts and direct sums}} \mathcal{SC}_2 \xrightarrow{\text{Karoubi envelope}} \mathcal{SC}$$

In general, the Karoubi envelope is the category which formally includes all “summands”, where a summand of an object M is identified by an idempotent $p \in \text{End}(M)$ corresponding to projection to that summand. Since $R\text{-molf}_{\mathbb{Z}}\text{-}R$ is abelian, it is idempotent-closed, and the Karoubi envelope of \mathcal{SC}_2 can be realized as a subcategory in $R\text{-molf}_{\mathbb{Z}}\text{-}R$. We refer the reader to Wikipedia and other easily found online sources for basic information about Karoubi envelopes.

This final category \mathcal{SC} (the Soergel category) is a \mathbb{k} -linear additive monoidal category with the Krull-Schmidt property. Soergel showed that, when \mathbb{k} is an infinite field of characteristic other than 2, the indecomposable bimodules in this category are enumerated by elements of the Weyl group and grading shifts (Theorem 6.16 in [42]). They are denoted by $B_w\{j\}$ for $w \in \mathcal{W}$ and $j \in \mathbb{Z}$. An indecomposable B_w is determined by the condition that it appears as a direct summand of $B_{\underline{i}}$, where $\underline{i} = i_1 \dots i_d$ and $s_{i_1} \dots s_{i_d}$ is a reduced presentation of w , and does not appear as direct summand of any $B_{\underline{i}}$, for sequences \underline{i} of length less than $d = l(w)$.

The Grothendieck ring of \mathcal{SC} is isomorphic to \mathcal{H} , under an isomorphism sending product bimodule $B_{\underline{i}}$ to product element $b_{\underline{i}}$ and $R\{1\}$ to t . Relations (2.1)–(2.3) on b_i ’s lift to isomorphisms of graded bimodules

$$(2.4) \quad B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\},$$

$$(2.5) \quad B_i \otimes B_j \cong B_j \otimes B_i \text{ for } |i - j| \geq 2,$$

$$(2.6) \quad (B_i \otimes B_{i+1} \otimes B_i) \oplus B_{i+1} \cong (B_{i+1} \otimes B_i \otimes B_{i+1}) \oplus B_i.$$

The first isomorphism is given by sending $1 \otimes 1$ in B_i to either $1 \otimes 1 \otimes 1$ or $1 \otimes x_i \otimes 1$ in $B_i \otimes B_i$. These two tensors were observed in the previous section to generate $B_i \otimes B_i$ as an R -bimodule. The second and third isomorphisms come from the following isomorphisms in $R\text{-molf}_{\mathbb{Z}}\text{-}R$:

$$(2.7) \quad B_i \otimes B_j \cong R \otimes_{i,j} R\{-2\} \cong B_j \otimes B_i \text{ for } |i - j| \geq 2$$

$$(2.8) \quad B_i \otimes B_{i+1} \otimes B_i \cong B_i \oplus (R \otimes_{i,i+1} R\{-3\})$$

$$(2.9) \quad B_{i+1} \otimes B_i \otimes B_{i+1} \cong B_{i+1} \oplus (R \otimes_{i,i+1} R\{-3\}).$$

In the category \mathcal{SC}_1 , given any two sequences \underline{i} and \underline{j} , the space $X = \text{Hom}_{\mathcal{SC}_1}(B_{\underline{i}}, B_{\underline{j}})$ is a graded free left R -module. Hence we can define its *graded rank*, an element of $\mathbb{Z}[t, t^{-1}]$, by choosing a homogeneous R -basis $\{y_j\}$ of X and letting $\text{grk} X \stackrel{\text{def}}{=} \sum_j t^{\deg y_j}$.

We can extend this assignment all the way to \mathcal{SC} , where we get a map

$$(\cdot, \cdot): \text{Ob}(\mathcal{SC}) \times \text{Ob}(\mathcal{SC}) \rightarrow \mathbb{Z}[t, t^{-1}]$$

by taking $M \times N$ to the graded rank of

$$\bigoplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{SC}}(M\{j\}, N),$$

where the latter obviously has the structure of a graded free left R -module. Shifting the grading of N will shift the grading of this Hom space in the same direction, while shifting the grading of M will shift the grading of the Hom space in the opposite direction. Hence this map descends to a t -semilinear product on the Grothendieck ring of \mathcal{SC} ,

$$K_0(\mathcal{SC}) \times K_0(\mathcal{SC}) \longrightarrow \mathbb{Z}[t, t^{-1}], \quad ([M], [N]) \stackrel{\text{def}}{=} \text{grk}(\bigoplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{SC}}(M\{j\}, N)).$$

Unsurprisingly, this is the same semilinear product as defined in Section 2.1. To check this, following Remark 2.1, we only need to show two things. The first is that B_i self-biadjoint, i.e. that $\text{Hom}_{\mathcal{SC}_1}(M \otimes B_i, N) = \text{Hom}_{\mathcal{SC}_1}(M, N \otimes B_i)$ and $\text{Hom}_{\mathcal{SC}_1}(B_i \otimes M, N) = \text{Hom}_{\mathcal{SC}_1}(M, B_i \otimes N)$ via some canonical adjointness maps. This is known, and will become evident in the sequel. The second is that when \underline{i} is an increasing sequence, $\text{Hom}_{\mathcal{SC}_1}(R, B_{\underline{i}})$ is a free left R -module of rank 1, generated by a morphism of degree d , the length of \underline{i} . This is a simple argument, presented below.

An R -bimodule map from R to B_i is clearly determined by an element of B_i on which right and left multiplication by polynomials in R are identical. Any element of B_i is of the form $v = f \otimes 1 + g \otimes x_i$, and clearly $x_j v = v x_j$ for $j \neq i, i+1$, and $(x_i + x_{i+1})v = v(x_i + x_{i+1})$. Hence v can be the image of 1 under a bimodule map from R if and only if $x_i v = v x_i$.

$$v x_i = f \otimes x_i + g \otimes x_i^2 = f \otimes x_i + g(x_i + x_{i+1}) \otimes x_i - g(x_i x_{i+1}) \otimes 1.$$

Thus v can be an image iff $g(x_i x_{i+1}) = f x_i$ and $f + g(x_i + x_{i+1}) = g x_i$, iff $f = -g x_{i+1}$. Such elements form a left R -module generated by the case $g = 1$, $f = -x_{i+1}$, or in other words by $v = 1 \otimes x_i - x_{i+1} \otimes 1$. The element v has degree 1 in B_i , so we deduce that $(R, B_i) = t$. Let us call φ_i the corresponding map $R \rightarrow B_i$, $1 \mapsto v$.

One may use the same argument for the general case. Suppose \underline{i} is increasing and has length d . $B_{\underline{i}}$ is a free left R -module of degree 2^d , with generators $\{1 \otimes_{i_1} f_1 \cdots \otimes_{i_k} f_k\}$ ranging over terms where either $f_l = 1$ or $f_l = x_{i_l}$. As an exercise, the reader may find

the criteria for a general element to be ad-invariant under x_i , and verify that the only possible bimodule maps $R \rightarrow B_{\underline{i}}$ are R -multiples of the following iterated version of φ_i :

$$R \rightarrow B_{i_1} \cong B_{i_1} \otimes R \rightarrow B_{i_1} \otimes B_{i_2} \cong \dots$$

The first map is φ_{i_1} , the second map is $\text{Id} \otimes \varphi_{i_2}$, and so forth. This generator is a map of degree d , so that $\text{Hom}_{\mathcal{SC}_1}(R, B_{\underline{i}}) = R\{d\}$ and $([R], [B_{\underline{i}}]) = t^d$.

It is worth reiterating here the dependence of the above story on the choice of \mathbb{k} . It is clear that when $\mathcal{W} = S_{n+1}$, our definitions of B_i above will make sense over any commutative ring \mathbb{k} , so we can define the analog of category \mathcal{SC}_1 for any \mathbb{k} . One can still define the isomorphisms that descend to the Hecke relations and the maps which correspond to the self-adjunction of B_i over \mathbb{k} , and so we can calculate Hom spaces as above. In particular, the graded rank of $\text{Hom}(B_{\underline{i}}, B_{\underline{j}})$ will be $(b_{\underline{i}}, b_{\underline{j}})$. However, the size of the Grothendieck group of the Karoubi envelope of \mathcal{SC}_1 is not known for an arbitrary commutative ring \mathbb{k} . Specializing to \mathbb{k} an infinite field of characteristic $\neq 2$, we may use Soergel's result [42, Satz 6.16] to state that the indecomposables in the Karoubi envelope correspond to elements of \mathcal{W} , and that \mathcal{SC} categorifies the Hecke algebra. When $\mathbb{k} = \mathbb{C}$ we can go even further and match $[B_w]$ with the Kazhdan-Lusztig basis element C'_w .

2.4. Categorification dictionary. Let us summarize the Soergel categorification of the Hecke algebra \mathcal{H} of the symmetric group S_{n+1} and the various structures on it, following [39, 40, 27, 32, 36] and assuming $\mathbb{k} = \mathbb{C}$.

The ring $\mathbb{Z}[t, t^{-1}]$ is canonically isomorphic to the Grothendieck group of the category $R\text{-fmod}$ of finitely-generated graded free R -modules and grading-preserving homomorphism. Under this isomorphism

$$K_0(R\text{-fmod}) \cong \mathbb{Z}[t, t^{-1}]$$

$[R]$ goes to 1 and $[R\{d\}]$ to t^d . The bar involution on $\mathbb{Z}[t, t^{-1}]$ lifts to the contravariant equivalence that takes $M \in R\text{-fmod}$ to $\text{Hom}_R(M, R)$, the latter naturally viewed as an R -module.

The Hecke algebra \mathcal{H} is canonically isomorphic to $K_0(\mathcal{SC})$, the Grothendieck group of \mathcal{SC} . Multiplication by t operator corresponds to the grading shift: $[M\{d\}] = t^d[M]$. Multiplication in the Hecke algebra corresponds to the tensor product of bimodules:

$$[M] \cdot [N] := [M \otimes_R N].$$

The isomorphism takes $[B_i]$ to b_i and $[B_{\underline{i}}]$ to $b_{\underline{i}}$.

The Kazhdan-Lusztig basis $\{C'_w\}$ is the basis of balanced indecomposable objects $\{[B_w]\}$ of the Grothendieck group, $C'_w = [B_w]$. The pairing (x, y) becomes the graded rank of $\bigoplus_{j \in \mathbb{Z}} \text{Hom}(M\{j\}, N)$, viewed as (either left or right) free R -module:

$$([M], [N]) = \text{grk}\left(\bigoplus_{j \in \mathbb{Z}} \text{Hom}(M\{j\}, N)\right).$$

For a Soergel bimodule M the space of bimodule homomorphisms $\text{Hom}_{\mathcal{SC}_1}(R, M)$ is just the 0-th Hochschild cohomology $\text{HH}^0(R, M)$ of M . Thus, $\varepsilon(x)$ becomes the graded

rank of 0-th Hochschild cohomology of M , viewed as a free graded R -module:

$$\varepsilon([M]) := \text{grk}(\text{HH}^0(R, M)).$$

The bar involution on \mathcal{H} corresponds to the contravariant duality on the Soergel category that takes M to $\text{Hom}_{R'}(M, R')\{n(n+1)\}$:

$$\overline{[M]} = [\text{Hom}_{R'}(M, R')\{n(n+1)\}] = q^{n(n+1)}[\text{Hom}_{R'}(M, R')].$$

Here $R' := R^{S_{n+1}}$ is the subring of S_{n+1} -invariants in R , and the expression inside the square brackets on the right hand side is naturally an R -bimodule.

Antiinvolution ω lifts to the antiequivalence Ω of \mathcal{SC} that takes a bimodule to the biadjoint bimodule. For each bimodule M in \mathcal{SC} there exists a bimodule $\Omega(M)$ such that the functor of tensoring with $\Omega(M)$ is left and right adjoint (biadjoint) to the functor of tensoring with M . Bimodule $\Omega(M)$ is well-defined up to isomorphism. A homomorphism $f : M \rightarrow N$ of bimodules dualizes to the homomorphism $\Omega(f) : \Omega(N) \rightarrow \Omega(M)$, resulting in an antiequivalence of \mathcal{SC} . Notice that Ω takes B_i to itself and $B_{\underline{i}}$ to $B_{\underline{j}}$ where \underline{j} is given by reading \underline{i} from right to left.

The categorification of map τ is the functor $\text{HH}_0 : \mathcal{SC} \rightarrow R\text{-fmod}$ of taking the 0-th Hochschild homology $\text{HH}_0(M)$ of a Soergel bimodule. In particular, $\tau([M]) = \text{grk}(\text{HH}_0(M))$, and the equation $\tau(xy) = \tau(yx)$ categorifies to an isomorphism

$$\text{HH}_0(M \otimes_R N) \cong \text{HH}_0(N \otimes_R M)$$

of R' -modules.

Hecke algebra \mathcal{H} has a trace more sophisticated than ε or τ , called the Ocneanu trace [17], which describes the HOMFLY-PT polynomial. The categorification of the Ocneanu trace utilizes all Hochschild homology groups rather than just HH_0 , see [21, 44, 45].

Invertible elements T_i that satisfy the braid relations become [36] invertible complexes

$$0 \rightarrow R \otimes_i R \rightarrow R \rightarrow 0$$

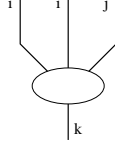
in the homotopy category of the Soergel category (bimodule R sits in cohomological degree 1). Their inverses T_i^{-1} become inverse complexes

$$0 \rightarrow R \rightarrow R \otimes_i R\{-2\} \rightarrow 0$$

with R in cohomological degree -1 . Homomorphism of the braid group into the Hecke algebra is categorified by a projective functor from the category of braid cobordisms between $(n+1)$ -stranded braids to the category of endofunctors of the homotopy category of the Soergel category [24].

2.5. Diagrammatic calculus for bimodule maps. We follow the standard rules for the diagrammatic calculus of bimodules, or more generally for the diagrammatic calculus of a monoidal category. The R -bimodule B_i is denoted by a point labelled i placed on a horizontal line. The tensor product of bimodules is depicted by a sequence of labelled points on a horizontal line, so that tensor products are formed “horizontally”. A vertical line labelled i denotes the identity endomorphism of B_i . Labelled

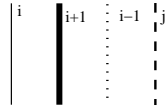
lines placed side by side denote the identity endomorphism of the tensor product of the corresponding bimodules. More general bimodule maps are represented by some symbols connecting the appropriate lines, and are composed “vertically”, and tensored “horizontally”. All diagrams are read from bottom to top, so that the following diagram may represent a bimodule map from B_k to $B_i \otimes B_i \otimes B_j$.



The horizontal line without marked points represents R as a bimodule over itself. The empty planar diagram represent the identity endomorphism of this bimodule. Planar diagrams without top and bottom endpoints (without boundary) represent more general endomorphisms of R . We generally do not draw the horizontal line at all. We won't draw horizontal lines, only the marked points on them, which will appear as endpoints of planar diagrams.

The structure of bimodule categories (or more generally strict 2-categories) guarantees that a planar diagram will unambiguously denote a morphism of bimodules.

We will be using so many such pictures that it will become cumbersome to continuously label each line by an index. Generally, the calculations we do will work independently for each i , and can be expressed with diagrams that use lines labelled i , $i + 1$ and the like. In these circumstances, when there is no ambiguity, we will fix an index i and draw a line labelled i with one style, a line labelled $i + 1$ with a different style, and so forth, maintaining the same conventions throughout the paper. We use different styles of lines because most printers are black and white, but we recommend that you do your calculations at home in colored pen or pencils instead; we even refer to the labels as “colors” throughout this paper.



We use the styles above when referring to indices i , $i + 1$, $i - 1$, and j , where j will be used unambiguously for any index which is “far away” from any other indices in the picture (in other words, when drawing a picture only involving i -colored strands, we require $|i - j| > 1$, while for a picture involving both i and $i + 1$ we require $j < i - 1$ or $j > i + 2$).

2.6. Methodology.

Proposition 2.2. *Suppose we choose the subset of the morphisms in \mathcal{SC}_1 , including the identity morphism of each object, as well as the following morphisms:*

- (1) *The generating morphism from R to B_i .*
- (2) *The isomorphisms that yield the Hecke algebra relations, as well as the respective projections to and inclusions from each summand in (2.6).*

(3) *The unit and counit of adjunction that make B_i into a self-adjoint bimodule.*

Consider \mathcal{C} the subcategory generated monoidally over the left action of R by these morphisms, i.e. it includes left R -linear combinations, compositions, and tensors of all its morphisms. Then \mathcal{C} is a full subcategory, and thus it is actually \mathcal{SC}_1 .

Proof. For any objects M, N in \mathcal{SC}_1 , there is an inclusion $\text{Hom}_{\mathcal{C}}(M, N) \subset \text{Hom}_{\mathcal{SC}_1}(M, N)$ of graded left R -modules (since it is clearly an inclusion of R -modules, and all generating morphisms are homogeneous). One can define grk for any graded left R -module M by choosing generators of $M/(R^+M)$, where R^+ is the ideal of positively graded elements, and it is a simple argument that a submodule of a free graded R -module with the same graded rank is in fact the entire module. So we need only show that Hom spaces in \mathcal{C} have the same graded rank.

We can define a bilinear form on the free algebra generated by b_i by the formula $(b_{\underline{i}}, b_{\underline{j}}) = \text{grk} \text{Hom}_{\mathcal{C}}(B_{\underline{i}}, B_{\underline{j}})$. Our direct sum decompositions imply that the Hecke algebra relations are in the kernel of this bilinear form, so it descends to a bilinear form on the Hecke algebra. Each b_i will be self-adjoint. When \underline{i} is increasing, $\text{Hom}_{\mathcal{C}}(R, B_{\underline{i}})$ contains the generator of the free rank one R -module $\text{Hom}_{\mathcal{SC}_1}(R, B_{\underline{i}})$, since that generator is the tensor of the generating morphisms from R to various B_i . Hence it is in fact the entire module, so $\text{Hom}_{\mathcal{C}}(R, B_{\underline{i}}) \cong R\{d\}$, and $(1, b_{\underline{i}}) = t^d$.

By unicity, this inner product agrees with our earlier inner product on the Hecke algebra. In particular, the graded ranks agree, and the inclusion is full. \square

Below we will construct a category \mathcal{DC}_1 of diagrams via generators and local relations, where the Hom spaces will be graded R -bimodules. We will construct a functor \mathcal{F}_1 from \mathcal{DC}_1 to \mathcal{SC}_1 , showing that our diagrams give graphical presentation of morphisms in \mathcal{SC}_1 . The morphisms in the image of \mathcal{F}_1 will include all the morphisms enumerated in Proposition 2.2, hence the functor will be full. Calculating the Hom spaces in \mathcal{DC}_1 between certain objects (corresponding to $R, B_{\underline{i}}$ for \underline{i} increasing), we may use a similar argument to the above proposition to show that they are free R -modules of the same graded rank as the Hom spaces in \mathcal{SC}_1 . Then the functor \mathcal{F}_1 will be faithful, and an equivalence of categories. This describes \mathcal{SC}_1 in terms of generators and relations.

Let \mathcal{DC}_2 be the category whose objects are finite direct sums of formal grading shifts of objects in \mathcal{DC}_1 , but whose morphisms only include degree 0 maps. Finally, let $\mathcal{DC} = \text{Kar}(\mathcal{DC}_2)$ be the Karoubi envelope of \mathcal{DC}_2 . The functor \mathcal{F}_1 lifts to functors \mathcal{F}_2 and \mathcal{F} , as in the picture below, with all three horizontal arrows being equivalences of categories.

$$\begin{array}{ccc}
\mathcal{DC}_1 & \xrightarrow{\mathcal{F}_1} & \mathcal{SC}_1 \\
\downarrow & & \downarrow \text{Grading shifts and direct sums} \\
\mathcal{DC}_2 & \xrightarrow{\mathcal{F}_2} & \mathcal{SC}_2 \\
\downarrow & & \downarrow \text{Karoubi envelope} \\
\mathcal{DC} & \xrightarrow{\mathcal{F}} & \mathcal{SC}
\end{array}
\tag{2.10}$$

Finally, if one draws the generating morphisms in \mathcal{DC}_1 using the appropriate pictures, the relations in \mathcal{DC}_1 enable us to view all morphisms unambiguously as graphs, instead of diagrams. Thinking of morphisms as graphs will be more convenient when we calculate Hom spaces.

To be very explicit, the diagrammatic definition of the category \mathcal{DC}_1 makes it clear how to define the functor \mathcal{F}_1 , since it is constructed out of local diagrams which are just depictions of morphisms in \mathcal{SC}_1 . The category would be entirely unchanged if one used different pictures for each generating morphism. However, when the “correct” pictures are chosen for the generators, the picture of any morphism turns out to be an isotopy invariant, and morphisms connecting multiple lines can be seen as vertices in a graph, so that with this choice we may define \mathcal{DC}_1 with graphs instead. It would be more difficult (a priori) to define the functor \mathcal{F}_1 directly from the graphical presentation, but that presentation is easier to examine, without relation to \mathcal{SC}_1 .

3. DEFINITION OF \mathcal{DC}



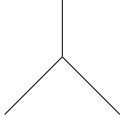
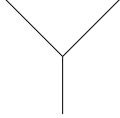

3.1. The category \mathcal{DC}_1 : one color. This section and the next several will hold the definition of the category \mathcal{DC}_1 , which will be a \mathbb{k} -linear additive monoidal category, with \mathbb{Z} -graded Hom spaces. Later it will become clear that Hom spaces are actually graded R -bimodules. It is generated monoidally by n objects $i = 1, \dots, n$, whose tensor products will be denoted $\underline{i} = i_1 \dots i_d$.

Morphisms will be given by (linear combinations of) diagrams inside the strip $\mathbb{R} \times [0, 1]$, constructed out of lines colored by an index i , and certain other planar diagrams, modulo local relations. The intersection of the diagram with $\mathbb{R} \times \{0\}$, called the *lower boundary*, will be a sequence \underline{i} of colored endpoints, the source of the map, and the *upper boundary* \underline{j} will be the target. A vertical line colored i represents the identity map from i to i . The monoidal structure consists of placing diagrams side by side, and composition consists of placing diagrams one above the other, in the standard fashion for diagrammatic categories.

We simultaneously define the functor $\mathcal{F}_1: \mathcal{DC}_1 \rightarrow \mathcal{SC}_1$. It is defined on objects by $\mathcal{F}_1(\underline{i}) = B_{\underline{i}}$, and on morphisms as below. To show that \mathcal{F}_1 is a functor one must check that the relations hold in \mathcal{SC}_1 , a series of simple but tedious calculations that is postponed to Section 5. All morphisms defined below will be homogeneous, with a specified degree, and \mathcal{F}_1 will preserve degrees of morphisms.

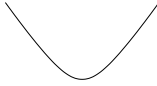
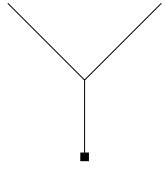

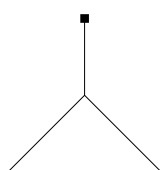
Instead of displaying all the generators and then all the relations, we will present them in an order based on the colors they use, for clarity of exposition. We will only display each generator with one color i , using the conventions described in Section 2.5, but the generator exists for each index i . In fact, it will turn out that the set of all relations is invariant under all color changes that preserve adjacency, under isotopy and rotation, and under horizontal and vertical flips!

The generating morphisms which use only one color (or fewer: the final generator is a map from \emptyset to \emptyset) are:

Symbol	Degree	Name	\mathcal{F}_1
	1	EndDot	$\begin{array}{c} fg \\ \uparrow \\ f \otimes g \end{array}$
	1	StartDot	$\begin{array}{c} x_i \otimes 1 - 1 \otimes x_{i+1} \\ \uparrow \\ 1 \end{array}$
	-1	Merge	$\begin{array}{cc} 0 & f \otimes g \\ \uparrow & \uparrow \\ f \otimes 1 \otimes g & f \otimes x_i \otimes g \end{array}$
	-1	Split	$\begin{array}{c} f \otimes 1 \otimes g \\ \uparrow \\ f \otimes g \end{array}$
	2	Mult by x_i	$\begin{array}{c} x_i \\ \uparrow \\ 1 \end{array}$

We give these maps names, but the names are very temporary. Once we explore the meaning of isotopy invariance, we will stop distinguishing between Merge and Split, and call them both *trivalent vertices*. Similarly we will stop distinguishing between StartDot and EndDot, and call them both *dots*.

We also use a shorthand for the following compositions:

Symbol	Degree	Name	\mathcal{F}_1
 $\stackrel{\text{def}}{=}$ 	0	Cup	$x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes x_{i+1}$ \uparrow 1
 $\stackrel{\text{def}}{=}$ 	0	Cap	$\begin{array}{c} 0 \\ \uparrow \\ f \otimes 1 \otimes g \end{array} \quad \begin{array}{c} fg \\ \uparrow \\ f \otimes x_i \otimes g \end{array}$

It is worth noting here that the maps indicated as the image of these generators under \mathcal{F}_1 are in fact R -bimodule maps. This is obvious for EndDot and Split. For Merge, one need only confirm that the only polynomials that can move across the tensors, on both top and bottom, are s_i -invariant. StartDot is precisely the generator φ_i of $\text{Hom}(R, B_i)$ discussed in Section 2.3.

We now list a series of relations using only one color, the *one-color relations*, dividing them into 4 sets for future reference. The first set are known as *polynomial slides*, which are obviously analogous to the definitions of the modules B_i .

$$(3.1) \quad \boxed{i} \Big| + \boxed{i+1} \Big| = \Big| \boxed{i} + \boxed{i+1} \Big|$$

$$(3.2) \quad \boxed{i} \boxed{i+1} \Big| = \Big| \boxed{i} \boxed{i+1} \Big|$$

$$(3.3) \quad \boxed{j} \Big| = \Big| \boxed{j} \Big|$$

Multiplication by x_i maps, for any i , are called *boxes*. A morphism from \emptyset to \emptyset consisting of a sum of products of boxes will be called a *polynomial*, and will obviously correspond to an element $f \in R$. As a shorthand, we may also draw such a morphism as a box with the corresponding element f inside. As a map from \emptyset to \emptyset , and thus a closed diagram, a polynomial may be placed in any region of another diagram.


The same obvious corollary as for B_i is that any polynomial can be “slid” across a line colored i , yielding a linear combination of morphisms with the lines unchanged, but with different configurations of boxes, and at most one box labelled i left behind.

$$(3.5) \quad \left| \begin{array}{c} f \end{array} \right| = \left| \begin{array}{c} P_i(f) \end{array} \right| + \left| \begin{array}{c} \partial_i f \end{array} \right| \left| \begin{array}{c} i \end{array} \right|$$

Relations (3.6)-(3.14) below state that i is a Frobenius object in the category \mathcal{DC}_1 . Existence of the functor \mathcal{F} implies that B_i is a Frobenius object in the category \mathcal{SC}_1 .

(3.7)

$$(3.9) \quad \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3}$$

(3.11) 

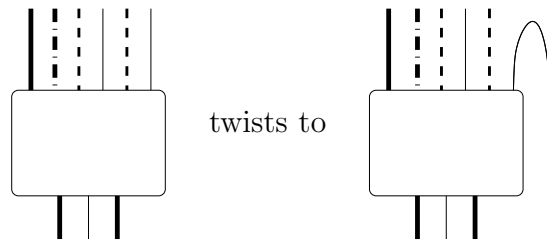
$$(3.12) \quad \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array} = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array}$$

$$(3.13) \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \blacksquare \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

$$(3.14) \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \blacksquare \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

Relations (3.6) and (3.7) are associativity of the Merge and coassociativity of the Split. The next two relations say that dot represents unit and counit morphisms relative to Merge and Split. Relation (3.10) says that the object i is self-adjoint, with cup and cap being biadjointness morphisms. Biadjunctions in all six categories in the commutative diagram (2.10) satisfy cyclicity conditions. For more on diagrammatics of biadjointness and the cyclicity property we refer the reader to [1, 2, 9, 25, 26]. In this language, notice that (Merge, Split) and (StartDot, EndDot) are mates under this adjunction, and are cyclic.

For simplicity of discussion, and ease of understanding for readers not versed in mates under adjunction, we will use the easily visualized notion of a *twist*. Given a morphism, one can twist it by taking a line which goes to the upper boundary and adding a cap, letting the line go to the other boundary instead. An example is given below.



One can also twist a downward line back up, or twist lines on the left as well. Two morphisms are twists of each other if they are related by a series of these simple twists, using cups and caps on the right and left side. For instance, relations (3.11) and (3.12) state that the Merge is a simple twist of the Split, for a variety of twists. If one applies the same twist to every term in a relation, one gets a twist of that relation. Because of biadjointness (3.10), twisting a line down and then back up will do nothing to the morphism. Once biadjointness is shown, all twists of a relation are equivalent, because twisting in the reverse direction we get the original relation back. When a morphism

has a total of m inputs and outputs, twisting a single strand will often be referred to as rotation by $\frac{180}{m}$ degrees.

The above relations imply that twisting any of the above generators by 360 degrees will do nothing. This is the notion of *cyclicity*, and it is precisely cyclicity in the presence of biadjoints which allows an isotopy class of a diagram to unambiguously represent a morphism. Moreover, Merge and Split are 60 rotations of each other, and each is invariant under 120 degree rotation, so we may represent them isotopy-unambiguously with pictures that satisfy the same properties. A similar statement holds for StartDot and EndDot, which is why we will refer to these morphisms as dots and trivalent vertices from now on.

We also may allow “isotopies” that retract dots back into lines, as in the definition of the cap or the cup. This particular “isotopy” always makes sense for diagrams representing Frobenius objects.

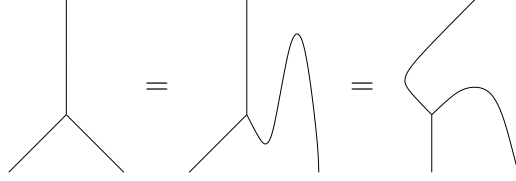
Henceforth, we don’t need to rigidly draw our diagrams with vertical lines only, but can draw lines freely, with the obvious convention that lines having boundary on the side of a diagram (instead of the top or bottom) are meant to be twisted upwards or downwards, and horizontal lines are meant to be turned into caps or cups in order that the picture represent an actual morphism in \mathcal{DC}_1 . In the diagrams above, we already took the liberty of bending vertical lines.

Associativity and coassociativity are twists of each other. This relation is written in a rotation-invariant form below, and shall be crucial in the sequel. We refer to all such relations, which permit one to “slide” a line over a trivalent vertex, as *associativity*.

$$(3.15) \quad \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}$$

One could draw the above diagram unambiguously (i.e. isotopy invariantly) as a 4-valent vertex (which is to say, the map is invariant under 90 degree twisting). We refer to either picture above as an “H”. Similarly, by repeated use of associativity, there is only one way for n strands to meet with $n - 2$ trivalent vertices, and it is invariant under the appropriate rotation, so we could replace it with an n -valent vertex. However, we will *not* make use of this simplification of drawings, for reasons that will become obvious in the sequel, and will use the term “4-valent vertex” for other purposes.

Note that relations (3.6), (3.10), (3.11), and (3.13) are sufficient to imply the other Frobenius relations, because of the remarks about twisting made above. Here is the proof of half of (3.12) using (3.11), as an illustrative example.



Now for the final set of one-color relations.

$$(3.16) \quad \begin{array}{|c|} \hline \blacksquare \\ \hline \blacksquare \\ \hline \end{array} = \boxed{i} \left| \right. - \left| \right. \boxed{i+I} = - \left| \right. \boxed{i+I} + \left| \right. \boxed{i}$$

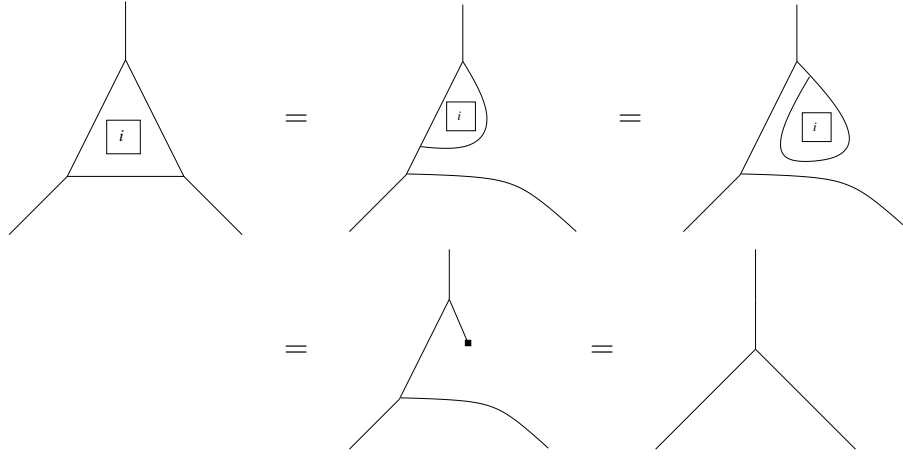
$$(3.17) \quad \begin{array}{|c|} \hline \blacksquare \\ \hline \blacksquare \\ \hline \end{array} = \boxed{i} - \boxed{i+I}$$

$$(3.18) \quad \begin{array}{c} \text{loop} \\ | \end{array} = 0 \quad \begin{array}{c} \boxed{i} \\ \text{loop} \\ | \end{array} = \begin{array}{|c|} \hline \blacksquare \\ \hline \end{array}$$

The second equality in (3.16) is just the relation (3.1). Relations (3.18) together with the earlier ones imply

$$(3.19) \quad \begin{array}{c} \text{loop} \\ | \end{array} = 0 \quad \begin{array}{c} \boxed{i} \\ \text{loop} \\ | \end{array} = \left| \right.$$

We refer to the relations (3.16)–(3.18) as *Dot and Needle relations*. They allow us to simplify diagrams with closed loops or paired dots. For example,



More generally

$$(3.20) \quad \text{Diagram of a pentagon with five external edges} = 0$$

$$(3.21) \quad \text{Diagram of a hexagon with six external edges and a box labeled 'i' inside} = \text{Diagram of a hexagon with six external edges and a box labeled 'i' outside}$$

We see that a region disjoint from the boundary with a polynomial f written in it simplifies as follows.

$$\text{Diagram of a hexagon with six external edges and a box labeled 'f' inside} = \text{Diagram of a hexagon with six external edges and a box labeled '\partial_i f' inside}$$

Therefore, any one-color diagram reduces to a planar forest (planar graph without cycles), with polynomials written in regions.

There is another relation which is equivalent (given the others) to the first equality in (3.16).

$$\begin{aligned}
(3.22) \quad & \begin{array}{c} | \\ | \end{array} = \begin{array}{c} | \quad \blacksquare \quad | \\ | \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \boxed{i} \end{array} - \begin{array}{c} \diagup \quad \diagdown \\ \boxed{i+1} \end{array} \\
& = \begin{array}{c} \diagup \quad \diagdown \\ \boxed{i} \end{array} - \begin{array}{c} \diagup \quad \diagdown \\ \boxed{i+1} \end{array}
\end{aligned}$$

This relation quickly leads to the decomposition $B_i \otimes B_i = B_i\{1\} \oplus B_i\{-1\}$, see Section 4.3.

For a single color and two variables x_1, x_2 , the category above, modulo the relation $x_1 = x_2$, is equivalent to the category considered by Libedinsky [29] in the case of a single label r . Morphisms given by dot, Merge, Split, and Cap correspond to morphisms $\widehat{\varepsilon}_r$, \widehat{m}_r , \widehat{p}_r , \widehat{j}_r and $\widehat{\alpha}_r$ in [29, Section 2.4]. Planar graphical notation, of paramount importance to us, is implicit in [29]. From here on, we diverge from Libedinsky's work, by generalizing to the case of the Weyl group S_{n+1} , while Libedinsky [29] investigates the right-angled case.

3.2. The category \mathcal{DC}_1 : adjacent colors. We now add some generators which mix adjacent colors, which we call *6-valent vertices*. Remember that the thick lines represent $i+1$, and the thin lines represent i .

Symbol	Degree	\mathcal{F}_1
	0	$ \begin{array}{ccc} 1 \otimes 1 \otimes 1 \otimes 1 & (x_i + x_{i+1}) \otimes 1 \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1 \otimes x_{i+2} \\ \uparrow & \uparrow \\ 1 \otimes 1 \otimes 1 \otimes 1 & 1 \otimes x_i \otimes 1 \otimes 1 \end{array} $
	0	$ \begin{array}{ccc} 1 \otimes 1 \otimes 1 \otimes 1 & 1 \otimes 1 \otimes 1 \otimes (x_{i+1} + x_{i+2}) - x_i \otimes 1 \otimes 1 \otimes 1 \\ \uparrow & \uparrow \\ 1 \otimes 1 \otimes 1 \otimes 1 & 1 \otimes x_{i+2} \otimes 1 \otimes 1 \end{array} $

Let us look at the first map above. The generating set $\{1 \otimes 1 \otimes 1 \otimes 1, 1 \otimes x_i \otimes 1 \otimes 1\}$ was chosen because x_i can be slid freely between the second and third slots. The image

of the second generator can be written more symmetrically as $(x_i + x_{i+1} + x_{i+2})(1 \otimes 1 \otimes 1 \otimes 1) - x_{i+2} \otimes 1 \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1 \otimes x_{i+2}$, where the first term is a polynomial symmetric in all the relevant variables, and thus can be slid anywhere. In the other two terms, x_{i+2} can not be slid freely under a line labelled $i+1$. In contrast, $1 \otimes x_{i+1} \otimes 1 \otimes 1$ and $1 \otimes 1 \otimes x_{i+1} \otimes 1$ are not equal, since x_{i+1} can not be slid over a line labelled $i+1$, but the images of both these elements are easier to remember, and are shown below.

$$\begin{array}{ccc}
 \begin{array}{c} \diagup \\ \diagdown \\ | \end{array} & \begin{array}{c} 1 \otimes 1 \otimes 1 \otimes x_{i+2} \\ \uparrow \\ 1 \otimes x_{i+1} \otimes 1 \otimes 1 \end{array} & \begin{array}{c} x_{i+2} \otimes 1 \otimes 1 \otimes 1 \\ \uparrow \\ 1 \otimes 1 \otimes x_{i+1} \otimes 1 \end{array} \\
 \begin{array}{c} \diagdown \\ \diagup \\ | \end{array} & \begin{array}{c} 1 \otimes 1 \otimes 1 \otimes x_i \\ \uparrow \\ 1 \otimes x_{i+1} \otimes 1 \otimes 1 \end{array} & \begin{array}{c} x_i \otimes 1 \otimes 1 \otimes 1 \\ \uparrow \\ 1 \otimes 1 \otimes x_{i+1} \otimes 1 \end{array}
 \end{array}$$

The way to remember this is that the variable which can't be slid is sent to the variable which can't be slid, from the middle on one side to the exterior on the other. Deriving this from the above definition is an exercise for the reader.

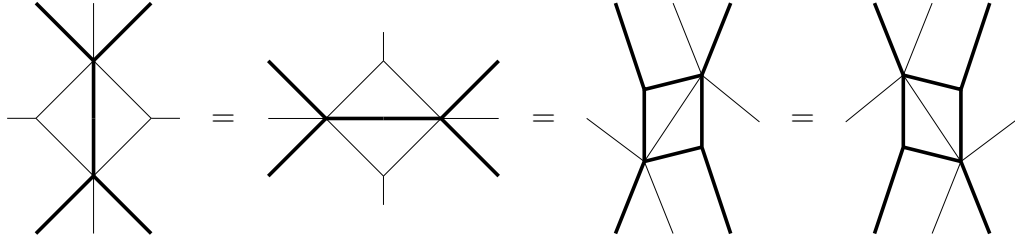
It will be shown in Section 4.3 that these are in fact R -bimodule maps.

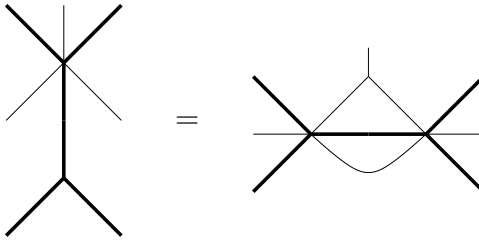
We list more relations below. For brevity we have only listed one color variant of the relations, but *we also impose the same relations with the colors switched!* The two color variants in general do not imply each other, and need to be checked separately.

$$(3.23) \quad \begin{array}{c} | \quad | \quad | \end{array} = \begin{array}{c} \diagup \\ \diagdown \\ | \end{array} - \begin{array}{c} \cap \\ | \\ \cup \end{array}$$

$$(3.24) \quad \begin{array}{c} \cap \\ | \\ \cup \end{array} = \begin{array}{c} \diagup \\ \diagdown \\ | \end{array} = \begin{array}{c} \cup \\ | \\ \cap \end{array}$$

$$(3.25) \quad \begin{array}{c} \diagup \\ \diagdown \\ | \end{array} = \begin{array}{c} \cap \\ | \\ \cup \end{array} + \begin{array}{c} \cup \\ | \\ \cap \end{array}$$

(3.26) 

(3.27) 

It will be shown in Section 4.3 that the first relation is related to the isomorphism $(B_i \otimes B_{i+1} \otimes B_i) \oplus B_{i+1} \cong (B_{i+1} \otimes B_i \otimes B_{i+1}) \oplus B_i$.

The second relation shows that our drawing of the 6-valent vertex is isotopy invariant, and isotopy classes of diagrams built out of our local generators will still unambiguously designate a morphism.

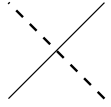
The relation (3.26) contains a number of equalities, and it is clear that the last equality is merely a rotation of the color switch of the first equality. As an exercise, one can show that any pair of equalities from (3.26) will imply both color variants of (3.27) as well as the remainder of the equalities from (3.26) (Hint: add a dot below the first equality in (3.26); use (3.27) twice inside the third picture in (3.26)).

One can observe that, together with the earlier relations, the relations (3.23) for both colors, (3.24) for one color, (3.25) for both colors, and the first equality in (3.26) for both colors together imply the remainder of the relations.

The important feature to notice is that the 6-valent vertex can be visualized as two trivalent vertices, one of each color, that overlap. A diagram as above with colors restricted to i and $i+1$ can be thought of as a planar graph with external (boundary) vertices of valence 1 and internal vertices of valence 1 (dots), 3 and 6. There are two types of vertices of valence 1 and 3 – one for each color, and one type of valence 6 vertex. Each of the two colors within forms a graph with univalent and trivalent internal vertices only. We call (3.27) and the first equality of (3.26) “double overlap associativity” relative to the thick subgraph (labelled $i+1$), because when viewed as an operation on the “thick”-colored graph, they are nothing more than the associativity relation for one color. Under the same transformations the “thin”-colored graph (labelled i) transforms differently, though. The last equality in (3.26) can be called “double overlap associativity” relative to the thin subgraph.

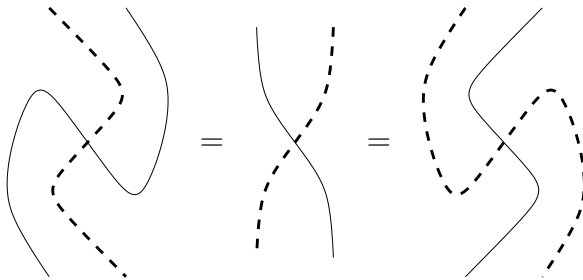
Notice that in the above relations internal regions of diagrams are free of boxes.

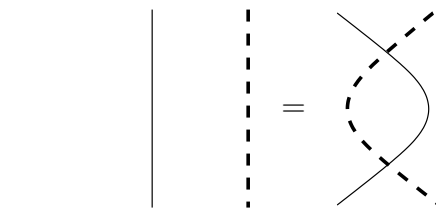
3.3. The category \mathcal{DC}_1 : distant colors. Fix j , an index which is not adjacent to i . In pictures involving both i and $i + 1$, we also assume j is not adjacent to $i + 1$. Remember that j is represented by a dashed line. This new generator is called a *4-valent vertex*, or a *crossing*.

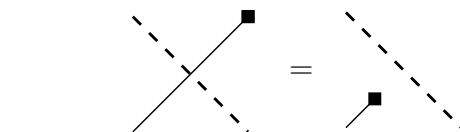
Symbol	Degree	\mathcal{F}_1
	0	$ \begin{array}{c} 1 \otimes 1 \otimes 1 \\ \uparrow \\ 1 \otimes 1 \otimes 1 \end{array} $

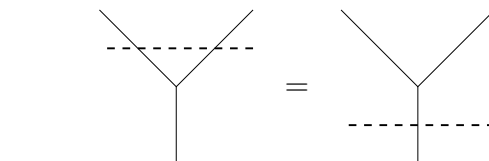
Note that this definition also covers the same picture with the colors reversed. The colors i and j can be switched freely since the only requirement was that they were distant from each other.

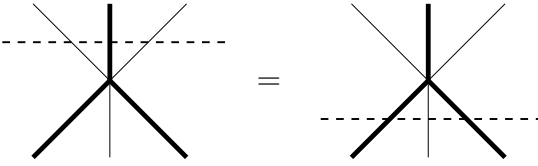
The fifth relation below holds when you switch i and $i + 1$, which does need to be checked separately (however, one color variant will follow quickly from the other by twisting and applying the first relation below). In the final relation, the new color represents an index k which is far away from both i and j ; this is the Reidemeister III move.

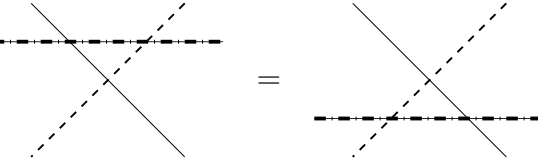
(3.28) 

(3.29) 

(3.30) 

(3.31) 

(3.32) 

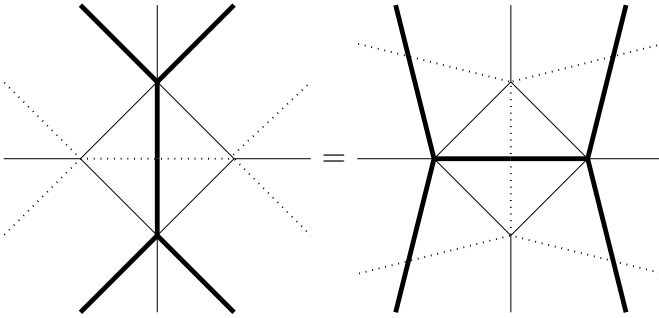
(3.33) 

All of the above relations are required (although only one color variant of (3.32) is needed to show the other). Other relations similar to (3.31) and (3.32), such as when j only goes through one or two lines of the 6-valent vertex, are easy to derive using those relations and (3.29). We will refer to (3.29) and (3.33) as the R2 and R3 moves respectively, because of the obvious analogy to knot theory. The R2 relation is essentially the isomorphism $B_i \otimes B_j \cong B_j \otimes B_i$, see Section 4.3.

The relations (3.29)-(3.33) imply that a j -colored strand can just be pulled underneath any morphism only using colors distant from j , whether it be a line, a dot, a trivalent vertex, or a 6-valent vertex.

Once again, the 4-valent vertices are drawn so that morphisms are isotopy invariant.

We have now listed all the generators of our subcategory: trivalent, 4-valent, and 6-valent vertices, and dots. There is one further relation, coming from the fact that $i + 1$ and $i - 1$ may not interact, but they do jointly interact with i . The final relation will be called “triple overlap associativity”:

(3.34) 

In the above diagram, dotted lines carry label $i - 1$, thick lines $i + 1$ and thin solid lines i . Rotating this relation by 90 degrees, we get the same relation except with $i + 1$ and $i - 1$ switched, so that only one variant is needed to imply both.

The “thick”-colored graph undergoes the associativity transformation. The same is true (symmetrically) with the “dotted”-colored graph. We are effectively performing double overlap associativity on colors i and $i + 1$, except with our third color $i - 1$ interfering in the only place it can.

This concludes the definition of the category \mathcal{DC}_1 and the functor \mathcal{F}_1 .

3.4. Graphical terminology. Because of the relations in \mathcal{DC}_1 , the morphism associated to a diagram is an isotopy invariant of the diagram, and the generators all satisfy some cyclicity relations. Moreover, the planar diagram generators can all be viewed as vertices in a graph with colored edges.

Definition 3.1. Henceforth, whenever we use the term *graph*, we are only considering a very special class of graphs. Graphs will all be planar graphs with boundary (that is, edges which go to the boundary of the plane), and each edge of the graph will be labelled with an integer from 1 to n , referred to as its “color.” We only allow univalent vertices (dots), trivalent vertices where each edge has the same color, 4-valent vertices where the colors entering alternate between i and j for $|i - j| \geq 2$, and 6-valent vertices where the colors entering alternate between i and $i + 1$. Finally, we allow edges which are not attached to the boundary or any vertex, which must necessarily form a circle.

Graphs will be often represented by the letter Γ . We may as well consider a graph to be embedded in the strip $\mathbb{R} \times [0, 1]$, just like our diagrams above, instead of the plane.

Definition 3.2. By an *isotopy* of graphs, we mean an isotopy of the graph as embedded in the strip, where we also allow isotopies to contract a dot into a line, or turn a line into a trivalent vertex attached to a dot, as in the definition of the cup and cap. These procedures are called *dot contraction* and *reverse dot contraction*.

A graph in the strip splits the strip into regions. The region on the far left will be called the *lefthand region*. The *righthand region* is defined similarly. The empty graph has a single region, which is both the lefthand and righthand region.

We may think of a graph as a representative (of an isotopy class of graphs modulo relations) of a morphism in \mathcal{DC}_1 , with all boxes ignored (i.e. ignoring the data of polynomials in each region). In particular, we could have defined the category \mathcal{DC}_1 to have morphisms which are isotopy classes of graphs with polynomials in each region, modulo the local relations (we no longer need the relations which merely state isotopy invariance). This is an equivalent definition. We will think of morphisms in \mathcal{DC}_1 as being graphs with polynomials henceforth.

Graphical notation is nice because cyclicity of our generating morphisms becomes inherent, and because planar graph theory can be used to aid some arguments. With this graphical terminology, Hom spaces in \mathcal{DC}_1 are enriched in $R\text{-molf}_{\mathbb{Z}}\text{-}R$, where the action of R on the left (resp. right) consists of placing a polynomial in the lefthand (resp. righthand) region, and the degree of a morphism is -1 for each trivalent vertex, 1 for each dot, 2 for each box, and 0 for each 6-valent vertex (this is preserved by dot contraction).

4. CONSEQUENCES

We now proceed to investigate the category \mathcal{DC}_1 with the ultimate goal of proving the following theorem.

Theorem 1. *The functor \mathcal{F}_1 from \mathcal{DC}_1 to \mathcal{SC}_1 is an equivalence of \mathbb{k} -linear monoidal categories with Hom spaces enriched in $R\text{-}\mathbf{molf}_{\mathbb{Z}}\text{-}R$.*

This theorem holds over any commutative ring \mathbb{k} . Assuming that \mathcal{F}_1 is in fact a functor, which will be verified in Section 5, we know as per the discussion in Section 2.6 that \mathcal{F}_1 is full, and need only verify that the Hom spaces in \mathcal{DC}_1 are free left (or right) graded R -modules of the appropriate rank. This chapter will focus on the consequences of our relations towards classifying all morphisms.

First, some terminology: Given a set S of graphs and a morphism φ in \mathcal{DC}_1 , we say that S *underlies* φ if φ can be written as a linear combination of morphisms, each of which is given by a graph $\Gamma \in S$ with polynomials in regions. We say that a graph Γ *reduces* to S if S underlies every morphism obtained from Γ by placing arbitrary polynomials in each region. Clearly reduction is transitive, in that if Γ reduces to S , and every graph in S reduces to S' , then Γ reduces to S' .

For instance, the relation (3.22) implies that if Γ is a graph with two adjacent edges of the same color, we may reduce it to the same graph with those two edges replaced by an “H”.

Definition 4.1. Let T be a subset of $\{1, \dots, n\}$. The T -*graph* of a graph will be the subgraph consisting of all edges colored i for $i \in T$. In particular, a vertex which is 6-valent with colors $i \in T$ and $i \pm 1 \notin T$ becomes a trivalent vertex for i , and a vertex which is 4-valent with colors $i \in T$ and $j \notin T$ is ignored, leaving a single i -colored edge in its wake. The T -graph is itself a graph by our above definition. More commonly, we will just consider the i -graph for a single color (i.e. $T = \{i\}$).

Our general method will be to derive *reduction moves*, which replace a graph with a set of graphs it reduces to, in order to determine a small set of graphs that every morphism reduces to. These reduction moves will often be determined by their action on the i -graph, since if we can simplify the i -graph for each color i , one at a time, then we can simplify the graph. The notion of reduction essentially ignores polynomials, so once we have found the graphs which underlie every morphism, we will investigate what happens when polynomials are placed in various regions.

Definition 4.2. We say a Soergel graph Γ is *simpler* than Γ' if one of the following conditions holds:

- (1) Γ has fewer connected components.
- (2) Γ has fewer 6-valent vertices, but the same number of connected components.
- (3) Γ has fewer trivalent vertices, but the same number of 6-valent vertices and connected components.

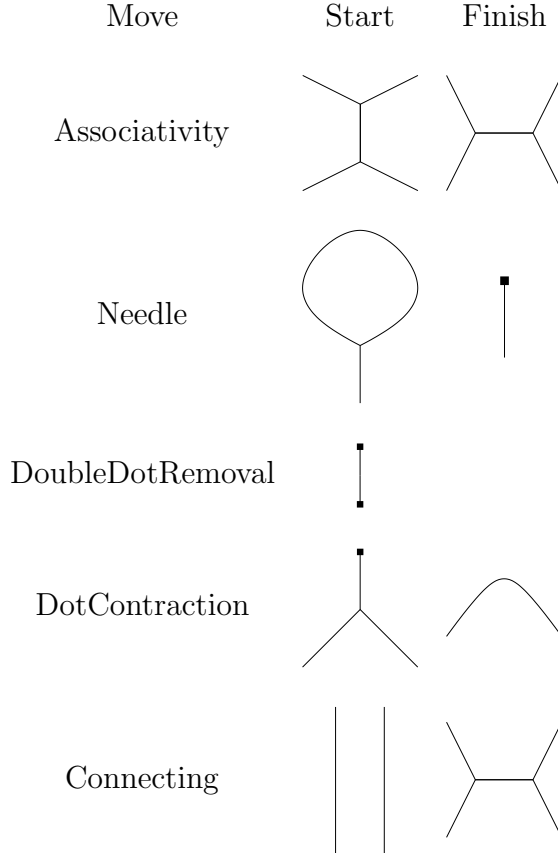
This is a partial order, clearly satisfying the descending chain condition, that will serve often as a base of induction.

We call a *boundary dot* any connected component of a graph which consists entirely of an edge starting at the boundary and ending in a dot.

We call a *double dot* any connected component of a graph which consists entirely of a edge with a dot on both ends.

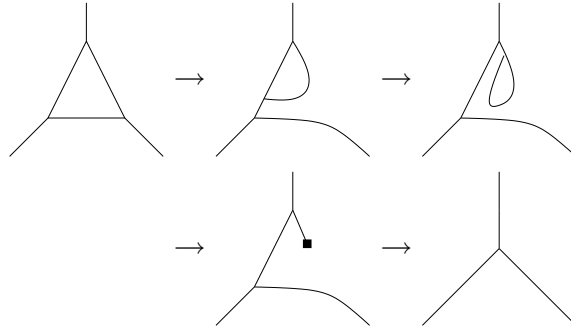
4.1. One color reductions. In this section, we assume all graphs consist of a single color.

First we discuss certain kinds of reduction moves on one color graphs, which always reduce a graph Γ to a single other graph Γ' . They take a subdiagram looking like Start, and replace that subdiagram with Finish. We allow Dot Contraction to work in both directions.

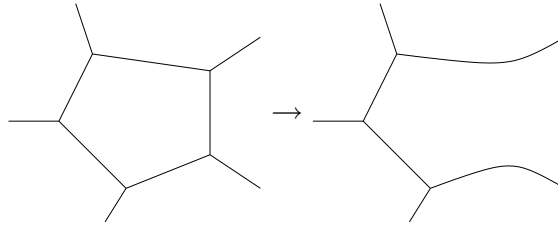


Remember, these are moves on graphs, not graphs with polynomials. Dot contraction is already included in the definition of isotopy, but it shows up here for a reason which shall be obvious soon. Note that the needle move, by adding a dot on the bottom, yields a reduction from the circle to a double dot. The only moves which change the connectivity of a graph are double dot removal, which deletes a connected component, and the connecting move, which has the potential to link two components into one.

An example of a reduction using these moves is shown below.



In fact, an inductive argument shows the following reduction, which should be familiar from the previous chapter, and only uses the associativity and needle moves.



This reduction immediately implies that any one-color graph reduces to a forest (a graph with no cycles or circles). The number of connected components is unchanged! A forest is composed out of a disjoint union of connected trees. Each dot present must be connected either to the boundary, another dot, or a trivalent vertex. Hence it is immediately obvious, using dot contraction and double dot removal, that we may assume each tree either has no dots, or is a boundary dot. In particular, when we make this assumption, we have removed each tree which had no boundary, and not changed the connectivity of trees touching the boundary. Just to be very precise, we use the following redundant definition.

Definition 4.3. A *simple tree* T with m boundary lines is the following (connected) graph with boundary:

- (1) If $m \geq 2$ then T is a trivalent tree with $m - 2$ vertices connecting all the boundary lines.
- (2) If $m = 1$ then T is a single boundary dot.
- (3) If $m = 0$ then T is the empty graph.

A *simple forest* is a forest, each of whose components is a simple tree.

Note that any two simple trees with m boundary lines are equivalent under the associativity move, so there is effectively one such tree. It is also easy to observe that when one applies the connecting move to edges from two disjoint simple trees, one gets a connected graph which is in fact a simple tree.

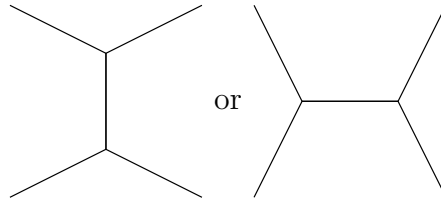
So we have proven the following useful proposition.

Proposition 4.4. *Using the associativity, needle, dot removal and dot contraction moves, one can reduce any one-color graph to a simple forest. All connected components of the original graph which did not have a boundary are removed, and after this is done, the connectedness is unchanged. Using the connecting move in addition, we can reduce the graph to a simple tree.*

In particular, a connected graph always reduces to a simple tree (which may be empty), without needing the use of the connecting move.

There are two useful sets of one-color graphs with m boundary lines, to which all others reduce. The first is just the simple tree, and the latter is the collection of all simple forests. The former is useful because there is a single graph, so we have fewer cases to deal with. The latter is useful because it doesn't require the connecting move. More importantly, these sets behave differently when we introduce polynomials into the equation. The following result will not be used in this paper, and is an immediate consequence of our calculation of Hom spaces, but is also an easy exercise. There are two useful spanning sets for the space of graphs with polynomials with m boundary lines. The first consists of the simple tree, with an arbitrary polynomial in the lefthand region (or rather, a basis for polynomials), and with either 1 or x_i in each other region. This spanning set is actually a basis. The second consists of the set of all forests, with an arbitrary polynomial in the lefthand region, and *no other polynomials!* Effectively, we may use relation (3.16) to replace x_i on one side of a line with x_{i+1} on the other side, at the cost of replacing the line with two dots in a third diagram. This lets us get from the first spanning set to the second. Relation (3.22) lets us go the other way. To show that the latter is not a basis, consider the three different ways to take a trivalent vertex and turn one edge into two dots: there is a relation between these and the trivalent vertex with a polynomial in the lefthand region.

4.2. i -colored moves. Now we demonstrate the procedure which allows us to simplify multicolor graphs. Let Γ be a graph, and fix a neighborhood in Γ whose i -colored graph is an “H”, that is, one of the two pictures below



Definition 4.5. Let A be the set of graphs Γ' which are identical to Γ outside the chosen neighborhood, and whose i -colored graph in that neighborhood is an “H” with the other picture. In other words, A is the set of all graphs where the associativity move has been locally applied to a specific “H”. Let S be the set of graphs simpler than Γ , using the definition of simpler found at the beginning of this section. If Y is a set of relations or reduction moves, we say that Y allows the *i -colored associativity move* to be applied to our chosen “H”, if we may reduce Γ to $A \cup S$ using Y . We say that Y allows the *strict i -colored associativity move* if we may reduce Γ to A .

We say that Y allows one to apply the *i-colored associativity move* to Γ if it can be done for every such neighborhood, i.e. for every instance of an *i-colored “H”* in the graph. Similarly for the strict version.

Definition 4.6. Similar definitions can be made for the strict and non-strict *i-colored double dot removal*, *i-colored dot contraction*, *i-colored needle move*, and *i-colored connecting move*. One chooses a neighborhood whose *i-colored* graph looks like the first column in the table of section 4.1, and asks if one can reduce the graph to pictures which locally look like the second column, possibly modulo simpler graphs.

All of the above are collectively called the *i-colored basic moves*.

Proposition 4.7. *The relations defining \mathcal{DC}_1 allow one to apply the i-colored basic moves to any graph Γ , so long as Γ does not contain either $i - 1$ or $i + 1$.*

The proof of this proposition is found in Section 5.2.

Remark 4.8. One may have wondered earlier why dot contraction was viewed as a reduction move, since it was built into the notion of isotopy. However, one can see that the *i-colored* dot contraction move does not merely follow from isotopy, since the *i-colored* trivalent vertex could actually be a 6-valent vertex in the full graph, and distant lines may cross over the picture.

The power of the proposition can be seen immediately:

Corollary 4.9. *Any graph without boundary Γ can be reduced to the empty graph.*

Proof. We induct on the simplicity of the graph and on the total number of colors present in the graph. By Proposition 4.7 we may apply the basic colored moves to the lowest color i . We may apply Proposition 4.4 to replace every connected component of the *i-graph* with the empty set. Hence, applying our reductions to Γ , we get a set of graphs which are either simpler than Γ , or do not include the color i . By the inductive hypothesis, each of these graphs reduces to the empty graph. \square

Note that we did not use the *i-colored* connecting move in this argument, since we dealt with each connected component separately. In general, it is unnecessary for any of the proofs in this paper.

One can apply a similar procedure to a graph with boundary. Namely, if i is the lowest (resp. highest) color, we can reduce the *i-graph* to a simple forest (or a simple tree). Then, within each region delimited by the *i-graph*, we can reduce the $i + 1$ -graph (resp. $i - 1$ -graph) to a simple forest (the number of boundary lines in each region will be determined by how many of the trivalent *i-graph* vertices actually come from 6-valent vertices). One can iterate the process.

Corollary 4.10. *Any graph whose boundary has at most one line of each color can be reduced to a disjoint union of boundary dots.*

Proof. We know we can reduce the *i-graph*, for i the lowest color present in the graph, to either the empty set or a boundary dot, depending on whether or not i appears in

the boundary. The dot need not be a boundary dot in the entire graph Γ , but it can encounter only 4-valent vertices on the way to the boundary. Since a dot can be slid under a 4-valent vertex, we may turn the dot into a boundary dot. Now, the remainder of the graph does not intersect the i -graph at all, so it can be reduced as though i were not present at all. Induction for the win. \square

4.3. \mathcal{F}_1 is fully faithful. Now, adding back polynomials into the equation, we have a classification of all morphisms, at least in the easy case.

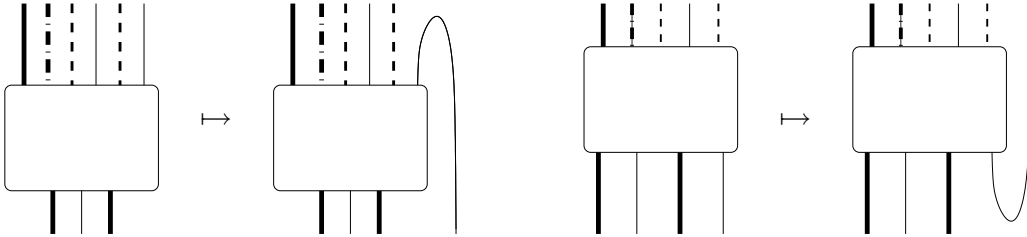
Corollary 4.11. *The space $\text{Hom}_{\mathcal{DC}_1}(\emptyset, \underline{i})$, where \underline{i} is a length d increasing sequence, is a free left (or right) R -module of rank 1, generated by a homogeneous morphism of degree d .*

Proof. By Corollary 4.10, the disjoint union of boundary dots underlies every morphism. This graph has a single region, so the only way to construct a morphism from it is to place an arbitrary polynomial in that region. The easiest proof that there are no further relations, i.e. that all such morphisms are generated freely over R by the morphism with the polynomial 1 in that region, is that this is easily observed to be true after application of the functor \mathcal{F}_1 , since adding a polynomial below the dots corresponds to left multiplication by R on $B_{\underline{i}}$, and $B_{\underline{i}}$ is a free left R -module. This generating morphism, which is nothing but d boundary dots, has degree d . \square

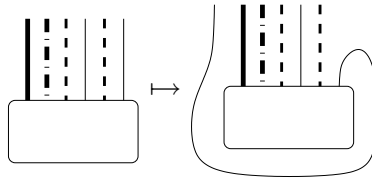
Moreover, as we have seen, \mathcal{F}_1 sends the boundary dot to the map φ_i from R to B_i , the free generator of $\text{Hom}_{\mathcal{SC}_1}(R, B_i)$, so that \mathcal{F}_1 yields an isomorphism $\text{Hom}_{\mathcal{DC}_1}(\emptyset, \underline{i}) \rightarrow \text{Hom}_{\mathcal{SC}_1}(R, B_{\underline{i}})$.

Corollary 4.12. *The object i in \mathcal{DC}_1 is self-biadjoint, so that there are natural isomorphisms $\text{Hom}_{\mathcal{DC}_1}(\underline{k}, \underline{j}) \rightarrow \text{Hom}_{\mathcal{DC}_1}(\underline{k}, i, \underline{j})$ and $\text{Hom}_{\mathcal{DC}_1}(\underline{k}, i, \underline{j}) \rightarrow \text{Hom}_{\mathcal{DC}_1}(i, \underline{k}, \underline{j})$. These compose when $\underline{k} = \emptyset$ to yield an isomorphism $\text{Hom}_{\mathcal{DC}_1}(\emptyset, \underline{j}) \rightarrow \text{Hom}_{\mathcal{DC}_1}(\emptyset, i, \underline{j})$.*

Proof. The first isomorphism and its inverse are shown below.



That these maps compose to be the identity is exactly the relation (3.10). The second isomorphism is effectively the mirror of this one. Here is a drawing of the third isomorphism.



□

At this point, one could construct a t -semilinear product on the free algebra generated by b_i , $i = 1, \dots, n$, via $(b_{\underline{i}}, b_{\underline{j}}) = \text{grkHom}_{\mathcal{DC}_1}(\underline{i}, \underline{j})$, and it would satisfy the three properties specified in Section 2.1. If we show that the Hecke algebra relations are in the kernel of this semilinear product then it will descend to the Hecke algebra, and by unicity will be exactly the semilinear product defined earlier. This will then tell us that the Hom spaces in \mathcal{DC}_1 are exactly what we wish them to be, and \mathcal{F}_1 is faithful and full. In fact, we will even get for free that $\text{Hom}(\underline{i}, \underline{j})$ is a free left R -module!

Essentially, what we want to do is look in \mathcal{DC}_2 , where we have direct sums and grading shifts, and prove the isomorphisms (2.7)-(2.9). Alternatively, we could look in the Karoubi envelope \mathcal{DC} , find idempotents corresponding to the auxilliary modules in (2.7) and friends, and show those isomorphisms. Finally, we could work entirely within \mathcal{DC}_1 and merely show the isomorphisms (2.6) after applying the Hom functor. For instance, showing that $\text{Hom}(ii, \underline{j}) = \text{Hom}(i, \underline{j})\{1\} \oplus \text{Hom}(i, \underline{j})\{-1\}$ will be sufficient. All these tactics are roughly the same.

The relation (3.22) precisely descends to $B_i \otimes B_i \cong B_i\{1\} \oplus B_i\{-1\}$. We decompose the identity of ii into the sum of two idempotents, and obtain orthogonal projections from ii to i of degrees 1 and -1 respectively.

$$(4.1) \quad \begin{array}{c} | \\ | \end{array} = \begin{array}{c} \alpha_1 \\ \text{---} \\ p_1 \end{array} \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \boxed{i} \end{array} - \begin{array}{c} \alpha_2 \\ \text{---} \\ p_2 \end{array} \begin{array}{c} \boxed{i+1} \\ \diagup \quad \diagdown \\ \text{---} \end{array}$$

Stated very explicitly, we have two projection maps $p_1, p_2: ii \rightarrow i$, and two inclusion maps $\alpha_1, \alpha_2: i \rightarrow ii$, as indicated in the diagram above, which is just relation (3.22) divided in half. Include the minus sign on the right picture into the map α_2 . Then one can quickly check $p_1\alpha_1 = 1_i$, $p_2\alpha_2 = 1_i$, $p_1\alpha_2 = 0$, $p_2\alpha_1 = 0$, and $1_{ii} = \alpha_1 p_1 + \alpha_2 p_2$. So these must be projections to and inclusions of summands, and they are maps of the correct degree.

Now we look at i and j which are not adjacent, which is the easy case. We have the isomorphism $B_i \otimes B_j \cong R \otimes_{R^{i,j}} R\{-2\} \cong B_j \otimes B_i$. It is clear that we may construct isomorphisms $\text{Hom}(ij, \underline{k}) \rightarrow \text{Hom}(ji, \underline{k})$ merely by precomposing with the appropriate 4-valent vertex, and (3.29) shows that these maps are inverses.

We may also extend our diagrammatic calculus to include additional modules from R -molf- R . If we allow a new color of line, call it w , to represent the bimodule $R \otimes_{R^{i,j}} R\{-2\}$, then we may define the following maps:

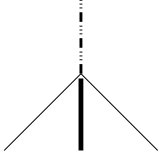
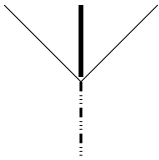
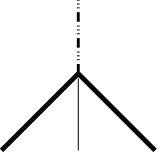
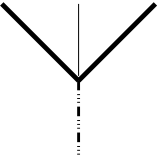
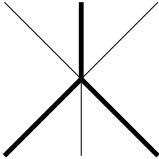
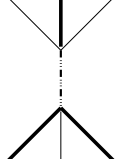
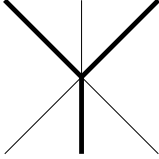
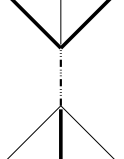
Symbol	Degree	Definition
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	0	$ \begin{array}{c} 1 \otimes 1 \\ \uparrow \\ 1 \otimes 1 \otimes 1 \end{array} $
	0	$ \begin{array}{c} 1 \otimes 1 \otimes 1 \\ \uparrow \\ 1 \otimes 1 \end{array} $

$$\begin{array}{c}
 \text{X-junction} \\
 \text{---} \\
 \text{Y-junction}
 \end{array}
 \stackrel{\text{def}}{=}$$

It is clear that composing these morphisms to get an endomorphism of ij will yield the identity map, and composing them to get an endomorphism of w will also yield the identity map. Thus we get isomorphisms $ij \cong w \cong ji$ in \mathcal{DC} . The bimodule that w represents is isomorphic to $B_{s_i s_j}$.

We now combine these techniques to deal with adjacent colors. We have isomorphisms $B_i \otimes B_{i+1} \otimes B_i \cong B_i \oplus (R \otimes_{R^{i,i+1}} R\{-3\})$ and $B_{i+1} \otimes B_i \otimes B_{i+1} \cong B_{i+1} \oplus (R \otimes_{R^{i,i+1}} R\{-3\})$. If we allow the new color of line, again called w , to represent the bimodule $R \otimes_{R^{i,i+1}} R\{-3\}$, then we may define the following maps:

Symbol	Degree	Definition
	0	$\begin{array}{ccc} 1 \otimes 1 & & (x_i + x_{i+1}) \otimes 1 - 1 \otimes x_{i+2} \\ \uparrow & & \uparrow \\ 1 \otimes 1 \otimes 1 \otimes 1 & & 1 \otimes x_i \otimes 1 \otimes 1 \end{array}$
	0	$\begin{array}{ccc} 1 \otimes 1 \otimes 1 \otimes 1 & & \\ \uparrow & & \\ 1 \otimes 1 & & \end{array}$
	0	$\begin{array}{ccc} 1 \otimes 1 & & 1 \otimes (x_{i+1} + x_{i+2}) - x_i \otimes 1 \\ \uparrow & & \uparrow \\ 1 \otimes 1 \otimes 1 \otimes 1 & & 1 \otimes x_{i+2} \otimes 1 \otimes 1 \end{array}$
	0	$\begin{array}{ccc} 1 \otimes 1 \otimes 1 \otimes 1 & & \\ \uparrow & & \\ 1 \otimes 1 & & \end{array}$
		$\stackrel{\text{def}}{=}$
		
		$\stackrel{\text{def}}{=}$
		

We call the first map ψ below.

Once again, composing the two maps that go through $i(i+1)i$ and w to get an endomorphism of w will yield the identity map of w , and the same is true respectively of $(i+1)i(i+1)$.

Then the equation (3.23), which is a decomposition of the identity of $i(i+1)i$, actually follows from this relation in \mathcal{DC} :

$$(4.2) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \alpha_1 \\ p_1 \end{array} - \begin{array}{c} \alpha_2 \\ p_2 \end{array}$$

There is a more explicit statement to be derived from this relation, completely analogous to the two line case, with projection and inclusion maps $p_1, \alpha_1, p_2, \alpha_2$ (include the minus sign on the right picture in α_2). Similarly we get a decomposition of $(i+1)i(i+1)$ via the same relation with the colors switched. The auxilliary module used here is in fact the indecomposable Soergel bimodule B_w where $w = s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

One can also think of w as the formal object in the Karoubi envelope that corresponds to the common “summand” of $i(i+1)i$ and $(i+1)i(i+1)$. The equations above actually represent a morphism in \mathcal{DC} , with the bimodule map being its image under the functor \mathcal{F} .

To state the result without extending the calculus to the Karoubi envelope, one may merely observe that there is a map $\text{Hom}(i(i+1)i, \underline{\mathbf{k}}) \rightarrow \text{Hom}((i+1)i(i+1), \underline{\mathbf{k}})$ and vice versa given by precomposing with the appropriate 6-valent vertex; there is another map $\text{Hom}(i(i+1)i, \underline{\mathbf{k}}) \rightarrow \text{Hom}(i, \underline{\mathbf{k}})$ given by precomposing with i_2 , and a map backwards given by precomposing with p_2 . Then (3.23) exactly yields that $\text{Hom}(i(i+1)i, \underline{\mathbf{k}}) \oplus \text{Hom}((i+1)i(i+1), \underline{\mathbf{k}}) \cong \text{Hom}((i+1)i(i+1), \underline{\mathbf{k}}) \oplus \text{Hom}(i, \underline{\mathbf{k}})$.

We would like to tie up a loose end and show that these auxilliary maps are actually R -bimodule maps, which would then imply that the 6-valent vertex is a bimodule map. To show this, one should calculate where ψ sends the element $f_1 \otimes f_2 \otimes f_3 \otimes f_4$, and show that one gets the same image after moving an s_i -invariant polynomial between f_1 and f_2 , or f_3 and f_4 , and after moving an s_{i+1} -invariant polynomial between f_2 and f_3 . In fact, the algorithm for computing ψ of this element is to slide f_2 to the left and f_3 to the right, leaving behind either 1 or x_{i+1} , and then computing the image. Effectively then, you need only compute this algorithm on $1 \otimes f \otimes g \otimes 1$, and show that you get the same answer after moving an s_{i+1} -invariant polynomial between f and g . This is a straightforward calculation, although it has so many terms that it shall not be shown here.

In conclusion, \mathcal{F}_1 is an equivalence of categories, and Theorem 1 is proven.

4.4. Color elimination. We have already shown that $\mathcal{DC}_1 \cong \mathcal{SC}_1$, without needing to investigate in depth any morphisms except those from \emptyset to $\underline{\mathbf{i}}$ an increasing sequence. We can deduce some additional facts about general morphisms which are not used elsewhere in this paper. These facts may come in handy when constructing the analogous category for arbitrary Coxeter systems. Proofs of the nontrivial propositions can be found in Section 5.

The question which naturally arises is whether or not the colors on the boundary determine which colors can appear in the graph. For $X \subset \{1, \dots, n\}$, (i.e. for a Coxeter subgraph of A_n) we let $\mathcal{DC}_1(X)$ be the category defined analogously but where edges can only be labelled by colors in X . If colors which don't appear on the boundary are not needed in the graph after reduction, then the natural inclusion of $\mathcal{DC}_1(X)$ into \mathcal{DC}_1 is fully faithful, as one would expect.

The case of edge colors is easy.

Corollary 4.13. *If i is the lowest (resp. highest) color in the graph and i does not appear in the boundary, then any graph can be reduced to a graph without that color.*

Proof. This follows immediately from the same proof as Corollary 4.9. \square

Corollary 4.14. *Any graph with only one color i on the boundary can be reduced to a graph solely of that color.*

Proof. Using the above, we may eliminate all other colors. \square

The general case is not as easy to prove graphically, but still holds true.

Proposition 4.15. *Let Γ be a graph where the color j does not appear in the boundary. Then Γ can be reduced to graphs not containing the color j at all.*

Proof. One can prove this proposition merely by counting the dimension of Hom spaces. We can think of any graph with boundary as a morphism in \mathcal{DC}_1 from \emptyset to \underline{j} . We have already “calculated” the space of such morphisms inductively. Namely, if \underline{j} has no repeated colors, then every morphism is a polynomial with boundary dots, and if \underline{j} has repeated colors, we apply biadjunction or idempotent decompositions to consider the Hom space as the sum of the Hom spaces of its summands. Since neither biadjunction nor idempotent decompositions adds any new colors to the graph, then we know that any graph must simplify to a morphism constructed purely out of the colors on its boundary! \square

It would be interesting to have a purely graphical proof of this result, similar to the other proofs which follow.

4.5. The e_1 quotient. As described in Section 2.2, one usually constructs Soergel bimodules with respect to the n -dimensional fundamental or geometric representation, instead of the $n + 1$ -dimensional standard representation. This amounts to working over the ring $R' = \mathbb{k}[x_1, x_2, \dots, x_{n+1}] / (x_1 + x_2 + \dots + x_{n+1})$, which is a quotient of R by the first elementary symmetric function e_1 . Since e_1 is symmetric, it is in the center of the category \mathcal{DC}_1 (it slides freely under all lines), and one may easily take the quotient category, setting $e_1 = 0$, without changing any of the diagrammatics. There is a functor from this quotient category to the appropriate category of Soergel bimodules in R' -molf- R' , which is an equivalence of categories, so this diagrammatical quotient category also categorifies the Hecke algebra.

One advantage to passing to the quotient $e_1 = 0$ is that, after inverting a suitable integer, we may remove the boxes from our list of generators. According to relation

(3.17), the double dot colored i is equal to $x_i - x_{i+1}$. Thus linear combinations of double dots of colors $i = 1, \dots, n$ will give us the \mathbb{k} -span of $x_1 - x_2, x_2 - x_3, \dots$ inside the space of linear polynomials. This span will not include x_i in R , which is why we require at least one box as an additional generator. However, it is easy to check that, if $n + 1$ is invertible in \mathbb{k} , then x_i is in the \mathbb{k} -span after passing to the quotient R' . As an example, when $n = 1$, the double dot is equal to $x_1 - x_2$, and passing to $x_1 + x_2 = 0$, we get $x_1 = \frac{1}{2}(x_1 - x_2)$. Thus, if $n + 1$ is invertible in \mathbb{k} , one can eliminate boxes from the quotient calculus altogether, replacing them with linear combinations of double dots.

5. PROOFS

5.1. \mathcal{F}_1 is a functor. We need now to check that each of our relations holds true for Soergel bimodules. It so happens that there are 28 relations that need to be checked independently, including color variations, and these 28 imply the remainder. These checks are completely straightforward and require little imagination. With a little imagination, however, there are some simple tricks that make checking almost all of these relations entirely trivial.

Recall that the necessary relations are:

- Three polynomial slides (3.1)–(3.3),
- Seven Frobenius relations: 2 equalities for biadjointness (3.10), 2 equalities twisting a trivalent vertex (3.11), 2 equalities twisting a dot (3.13), and one version of associativity (3.6),
- Three equalities for the Dot and Needle relations (equality (3.17) and first equalities in (3.16), (3.18)),
- Two color variants of the equation (3.23) for the idempotent decomposition of $i(i+1)i$ and $(i+1)i(i+1)$,
- Two equalities for twisting a 6-valent vertex (either two colors, or one color in both directions), see equation (3.24),
- Two color variants for adding a dot to a 6-valent vertex (equation (3.25)),
- Two color variants of the 90 degree rotation version of double overlap associativity (3.26), (3.27) which imply the remainder of the double overlap associativity relations,
- Six equalities with distant strands: the R2 and R3 moves (3.29), (3.33), one equality for twisting a 4-valent vertex (3.28), three equalities for pulling the distant-colored strand across a dot (3.30), trivalent vertex (3.31), and 6-valent vertex (3.32) respectively,
- Triple overlap associativity (3.34).

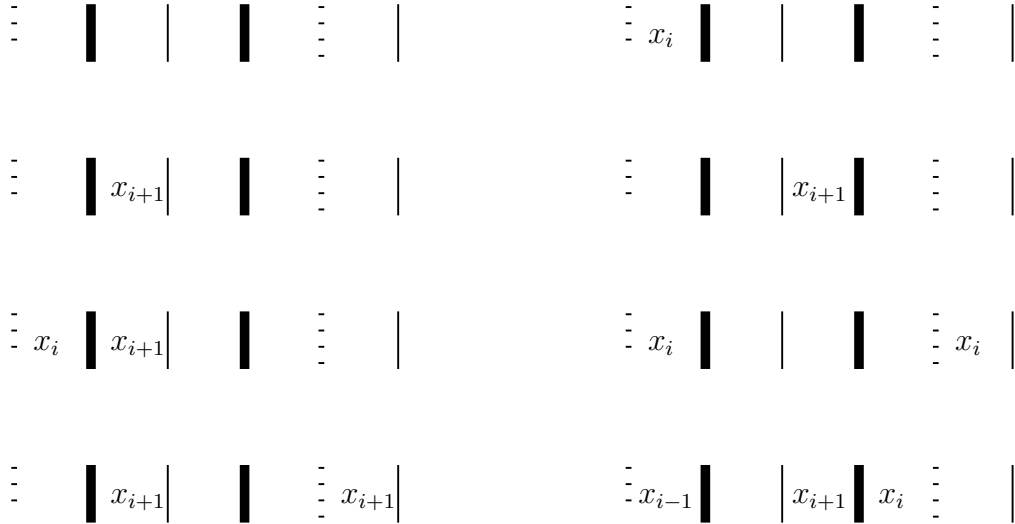
The polynomial slide relations obviously hold for Soergel bimodules.

For brevity, we will call any element which is a tensor product $1 \otimes 1 \otimes 1 \otimes \dots$ a *1-tensor*. EndDot, Split, and the 4-valent and 6-valent vertices all send 1-tensors to 1-tensors. This simple fact is already enough to prove relations (3.6), (3.29), (3.30), (3.31), and (3.33), since the bottom bimodule in each of these pictures is generated by the 1-tensor.

Caps and Merges kill a 1-tensor, while Cups and StartDots send it to a sum of two terms.

There are several choices to make when checking the various relations. Once the twisting relations are shown, one is free to prove any twist of the other relations. Also, we may choose which set of generators of the source bimodule to check equality on. Finally, whenever a Cup or a StartDot appears, there are two equivalent ways to write the result, and one might be easier than the other to evaluate.

An arbitrary module $B_{\underline{i}}$ of length d will have 2^{d-m} generators as an R -bimodule, where m is the number of different colors that appear. The reason for this is that between each pair of lines labelled i , consecutive if you ignore the other colors, a variable x_i (or alternatively x_{i+1}) might get stuck, and the number of such pairs is $d - m$. More explicitly, $B_{\underline{i}}$ will be generated by *any* 2^{d-m} linearly independent tensors corresponding to the power set of such pairs, where the tensor corresponding to a subset of pairs has either x_i or x_{i+1} somewhere between each pair (not counting possible variables corresponding to other pairs). It's a mouthful, but an example clarifies it. The following is an example of a set of generators for $B_{(i-1)(i+1)i(i+1)(i-1)i}$:

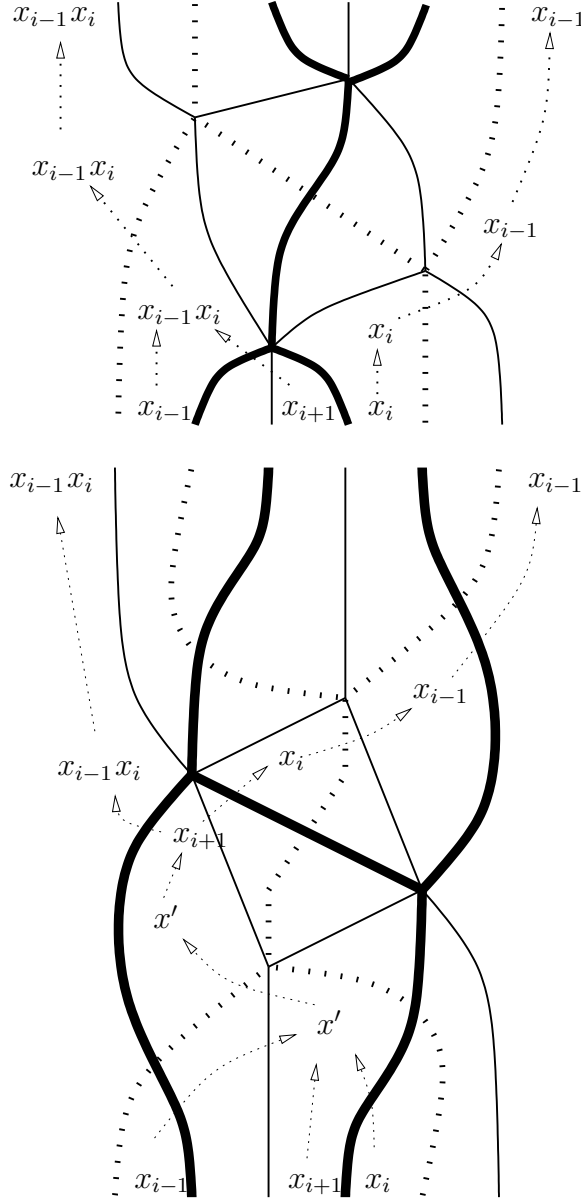


Each picture represents a tensor, where by convention a blank area is filled with a 1-tensor. Reading across, the second picture corresponds to the pair of $i - 1$ strands, the third to the pair of $i + 1$ strands, and the fourth to the pair of i strands. Since these three are linearly independent, they take care of all the generators with a linear term. Clearly the fourth picture could have also worked for the $i + 1$ strands, but if we had chosen it as such, we would have had to choose a new linearly independent vector for the i strands.

A clever choice of which generators to use may greatly simplify a calculation by reducing the number of terms in intermediate steps. There are two main instances when this occurs. Either 6-valent vertex with strands i and $i + 1$ will send $1 \otimes x_{i+1} \otimes 1 \otimes 1$ or $1 \otimes 1 \otimes x_{i+1} \otimes 1$ to a single tensor, while it may send $1 \otimes x_i \otimes 1 \otimes 1$ to the sum of

two tensors, thus doubling the work we need to do in the remainder of the calculation. Also, $x_i x_{i+1}$ entering a 6-valent vertex $i(i+1)i$ in the second slot may be moved across the i -strand to the left for a simple calculation, while x_i^2 leads to a more complicated solution.

In particular, the choice of generators above makes the verification of triple overlap associativity (3.34) rather straightforward. There is a nice graphical way of doing calculations, which we will demonstrate in perhaps the most difficult case, the highest degree generator.

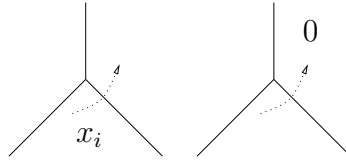


Here $x' = x_{i-1}x_ix_{i+1}$.

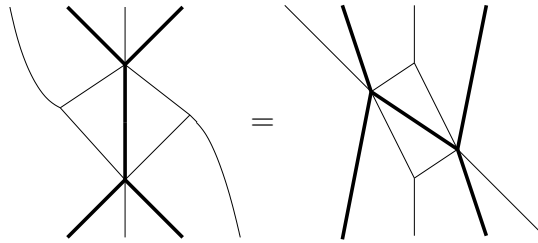
This graphical method is fairly self-explanatory. We keep track of the image at every stage in the calculation, which is one term as shown, with all blank spaces being filled by 1-tensors. An arrow indicates either bringing a symmetric polynomial through a line, or applying a 6-valent vertex to a tensor.

We leave it as an exercise to the reader to verify, using this graphical method of calculation, that both sides of the triple overlap associativity relation, as displayed above, send the other 7 generators to the same elements. Precisely, both sides send the 1-tensor to a 1-tensor, and the other generators to 1-tensors with polynomials in various slots: the 2nd and 4th generators to x_{i+2} in the last slot, the third generator to x_{i-1} in the last slot, the 5th to $x_{i-1}x_{i+2}$ in the last slot, the 6th to $x_i x_{i-1}$ in the first slot, the 7th to $x_{i+1}x_{i+2}$ in the first slot, and the 8th as shown above to $x_i x_{i-1}$ in the first slot and x_{i-1} in the last slot.

One can extend this graphical method easily to handle other morphisms. As an example, we show the Merge map below.



When x_{i+1} enters a Merge, it is sent to -1 . Similar pictures can be drawn for the Cap, and even simpler pictures can be drawn for the Split and EndDot, where no arrows are needed since 1-tensors are sent to 1-tensors. If you allow sums of such pictures, one can handle StartDot and Cup as well. With this method in place, checking (3.32) and both color variants of double overlap associativity using the following twist is particularly easy. Choose generators so that it is always x_{i+1} passed to a 6-valent vertex, and not x_i or x_{i+2} .



When there is a cup or a StartDot, this creates two terms, each of which must be evaluated separately. However, the terms can be written as $x_i \otimes 1 - 1 \otimes x_{i+1}$ or $1 \otimes x_i - x_{i+1} \otimes 1$, so we may choose the one whichever is more convenient. Often the problem of having two terms is very temporary. For example, consider what happens to the 1-tensor under the map



The cup creates two terms, the first with a 1 under the cap and the second with x_i under the cap. But the first term is annihilated immediately by the cap, and the cap eats the x_i from the other term, to return back a 1-tensor. So long as a cup or a StartDot appears right next to a cap or a merge, one of the two terms is always immediately annihilated. This is the case for (3.10), (3.11), (3.13), which all send 1-tensors to 1-tensors as a result. After a quick polynomial slide, the same is true for (3.28), and using the more convenient choice so the x_{i+1} ends up underneath the 6-valent vertex, a quick calculation shows it for (3.24) as well.

We have now given arguments or quick methods for almost all the relations. The remaining ones are trivially easy to check by hand. The rest of the one color relations can be checked on a single generator, the 1-tensor: (3.17) both sides send 1 to $x_i - x_{i+1}$; (3.16) both sides send the 1-tensor to $x_i \otimes 1 - 1 \otimes x_{i+1}$; the needle relation, where a 1-tensor is killed by a cap.

$$\begin{array}{c} | \\ \blacksquare \\ | \\ \blacksquare \\ | \end{array} = \boxed{i} \quad \bigg| \quad - \quad \bigg| \quad \boxed{i+I}$$

$$\begin{array}{c} \blacksquare \\ | \\ \blacksquare \end{array} = \boxed{i} - \boxed{i+I}$$

$$\begin{array}{c} \bigcirc \\ | \end{array} = 0$$

Only both color variants of (3.23) and (3.25) remain. These can be checked on the 1-tensor w and the generator $x \stackrel{\text{def}}{=} 1 \otimes x_{i+1} \otimes 1 \otimes 1$ regardless of whether we are in $B_{i(i+1)i}$ or $B_{(i+1)i(i+1)}$.

$$\begin{array}{c}
\text{Three vertical lines} = \text{Crossing of two lines} - \text{Crossing of two lines with dots} \\
\text{Crossing of two lines with dots} = \text{Crossing of two lines with dots} + \text{Crossing of two lines with dots} \\
\text{Crossing of two lines with dots} = \text{Crossing of two lines with dots} + \text{Crossing of two lines with dots}
\end{array}$$

Here $y = x_i$ for one color variant, $y = x_{i+2}$ for the other.

5.2. Graphical proofs. In this section, we provide graphical proofs for a series of propositions which state when the relations of \mathcal{DC}_1 allow us to perform various reduction moves. Recall that the *strict* i -colored needle move takes a graph Γ , whose i -graph in some neighborhood is a needle, and expresses Γ as a linear combination of graphs Γ' whose i -graph has a dot in that neighborhood instead. The (*nonstrict*) i -colored needle move is the same, except that the linear combination may also include graphs Γ' which are simpler than Γ , but have arbitrary i -graphs. We only require use of the nonstrict versions in this paper, but we prove the strict version when it occurs.

Because more than three colors may be required for some proofs, we use extra line styles that designate other arbitrary colors, and are very explicit about what colors they are allowed to be. A thin line will still always represent i and a thick line $i + 1$, but we will be lax about other colors.

Lemma 5.1. *One may (strictly) pull a distant line under any other graph, like in the relations (3.29)-(3.33).*

Proof. The only significant part of this lemma is that we can still apply the relations mentioned, even in the presence of arbitrary polynomials in each region. However, it is an immediate observation that any polynomial inside a region bounded by at least two lines of distant colors may be slid entirely out of the region, via a slide which does not affect the underlying graph. \square

Remark 5.2. In general, one must account for the existence of polynomials. For instance, the relation (3.23) does not immediately imply that we can apply the graph reduction which sends two 6-valent vertices to a combination of the lines and the “H” with dots, since the composition of the two 6-valent vertices might have x_i in an internal region, which can not be slid out. One can still effectively move a polynomial from one side of a line to the other, using relation (3.16), but this leaves an extra term where the line is replaced by a pair of dots.

Proposition 5.3. *(See Proposition 4.7) One may apply the strict i -colored double dot removal move to any graph Γ .*

Proof. Suppose the i -graph of Γ contains two dots connected by an edge. Then in Γ , the “edge” connecting them can only be a series of 4-valent vertices with various j -strands

for j distant from i . Since dots can be slid across distant-colored strands by (3.30), we may slide the dots until they are connected directly by an edge, and just use (3.17) to reduce the graph. \square

Proposition 5.4. *One may apply the i -colored dot contraction move to any graph Γ .*

Proof. Suppose the i -graph of Γ contains a trivalent vertex connected to a dot. In Γ , the trivalent vertex is either a trivalent or 6-valent vertex v . Again, in Γ the dot may be slid across distant-colored strands until it is connected directly to v . If v is trivalent then isotopy allows us to contract the dot, while if v is 6-valent, (3.25) sends Γ to the sum of two simpler graphs (note: one is the graph after dot contraction, but one is not, so we can not necessarily apply the strict dot contraction move!). \square

The following lemma is needed to prove the remainder of the basic i -colored moves.

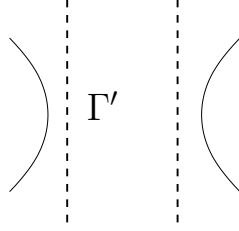
Lemma 5.5. *Let Γ be a sequence of lines \underline{j} , such that i does not appear. Then Γ is equal in \mathcal{DC}_1 to a sum of idempotents, such that each idempotent factors through a sequence of lines \underline{k} where $i-1$ and $i+1$ each appear no more than once, and such that the idempotents do not introduce any new colors not present in \underline{j} .*

Proof. This proof is akin to the combinatorial argument that the relations defining the Hecke algebra or the symmetric algebra are enough to take any complicated word in b_j or s_j to a reduced form. We use induction on the length of \underline{j} . If $k(k+1)k$ appears in \underline{j} for some k , then using (3.23), we may factor through $(k+1)k(k+1)$ instead, modulo morphisms that factor through shorter length sequences. If any color appears twice consecutively in \underline{j} then we may apply (3.22) to replace the two adjacent lines with an “H”, which factors through a sequence of lines of shorter length. If jk appears for j distant from k , then applying the R2 move we can factor through kj . None of these procedures added any new colors. So if we consider the sequence \underline{j} as a word in the symmetric group $s_{j_1}s_{j_2}\dots s_{j_d}$, then any non-reduced words will reduce to smaller length sequences, and any two reduced words for the same element will factor through each other, modulo smaller length sequences.

We may now use the easily observable fact that any element of the symmetric group can be represented by a reduced word which uses s_1 at most once, or s_n at most once (although not both!). The obvious corollary is that any element of the symmetric group which has a reduced word presentation that doesn't use s_i , can be given a reduced word presentation which uses s_{i-1} and s_{i+1} at most once (since all s_j for $j \geq i$ commute with all s_k for $k < i-1$, we can view the s_k as forming a word in S_{i-2} , and the s_j as forming a word in S_{n-i+1}). \square

Proposition 5.6. *One may apply the strict i -colored connecting move to any graph Γ , so long as Γ does not contain either $i-1$ or $i+1$.*

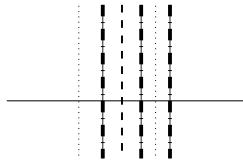
Proof. Assume $i-1$ does not appear, although the $i+1$ case is analogous. Induct on the number of colors present in the graph: the base case of one color is clear. Then we are attempting to apply the i -colored connecting move to some region of the graph which looks like this:



Γ' is some graph composed entirely of the colors $1, \dots, i-2$ and $i+1, \dots, n$. In fact, in some neighborhood Γ' is just a sequence of lines, and by the earlier lemma, we get a sum of graphs where, in a smaller neighborhood, this sequence contains at most a single $i+1$ strand (and no $i-1$ strands). For each term, apply the R2 move to bring the i -strands past all the other colors so that they are separated by either nothing or a single $i+1$ strand. We have not yet altered the i -graph at all. Then equation (3.22) or equation (3.23) (respectively) will allow the strict i -colored connecting move. \square

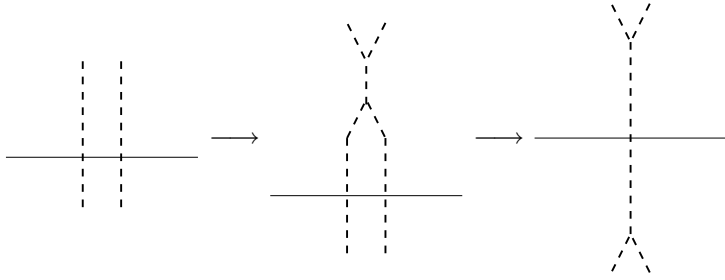
Corollary 5.7. *Suppose we are given a graph Γ and a neighborhood of an i -colored strand in Γ . Then Γ is equivalent in \mathcal{DC}_1 to a linear combination of graphs Γ' , which are equal to Γ outside the neighborhood, and has the same i -graph. The sequence of lines crossing the i -strand in Γ' contains $i+2$ and $i-2$ each at most once.*

Proof. Consider the neighborhood of an edge colored i in a Soergel graph Γ . Then in this neighborhood there may be a sequence \underline{j} of lines all distant from i which cross over the i -strand, but in this sequence $i, i-1$, and $i+1$ do not appear:



The horizontal line is i , the vertical lines are \underline{j} . According to the above lemma, we may look at \underline{j} right above where it hits the i -strand, and replace it with a sum of idempotents which factor through “nice” sequences where $i+2$ and $i-2$ each appear at most once. Consider each term in the sum: i is distant from every color appearing in the idempotent, so one can slide the i -strand past every morphism in the idempotent until it runs through the nice sequence of lines, and the rest of the idempotent is now outside of some smaller neighborhood of the i -strand in the new graph. Also, all polynomials can be moved outside the neighborhood as well.

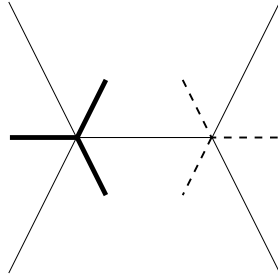
An illustration is given below.



□

Proposition 5.8. *One may apply the strict i -colored needle move to any graph Γ .*

Proposition 5.9. *One may apply the i -colored associativity move to any i -colored “ H ” in any graph Γ , so long as the “ H ” does not look like the following picture:*



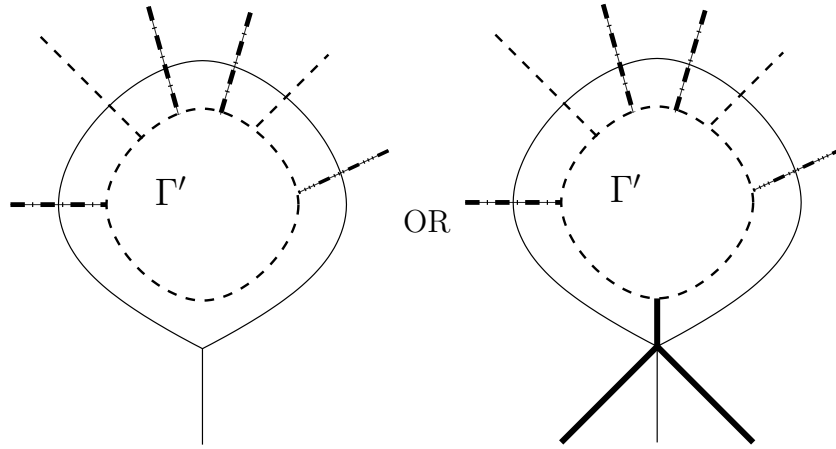
where the two vertices are 6-valent in Γ with different colors $i + 1$ and $i - 1$.

In particular, we may apply the i -colored associativity move to any graph Γ , so long as $i + 1$ or $i - 1$ is not present in the graph.

Proof. We apply induction on the number of colors present in the graph to prove these two propositions simultaneously. Both propositions hold when i is the only color in the graph, by (3.15), (3.18), polynomial slides, and (3.19), where the latter is needed because an arbitrary polynomial may be within the eye of the needle. The inductive hypothesis then implies, with the previous propositions, that for any graph with fewer colors, we may apply *all* the basic k -colored moves, when the graph does not contain either $k - 1$ or $k + 1$. In this case Proposition 4.4 says that we may reduce the k -graph to a simple tree or forest. Within each region of Γ delineated by the i -graph the color i is absent, so there are fewer total colors and we may reduce both the $i - 1$ -graph and the $i + 1$ -graph inside this region to a simple tree.

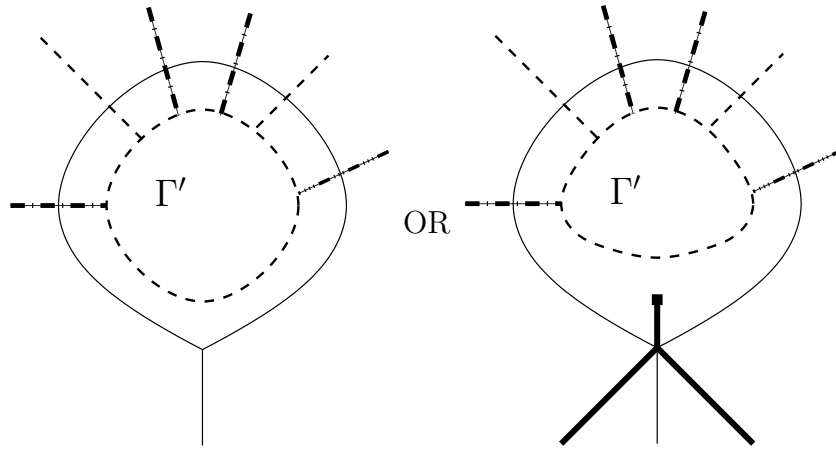
Needle move

So suppose the i -colored graph contains a needle. The trivalent vertex of the needle is either trivalent or 6-valent in Γ , and the edge of the needle may contain a series of 4-valent vertices. So a neighborhood of the needle in Γ looks like one of the pictures below.

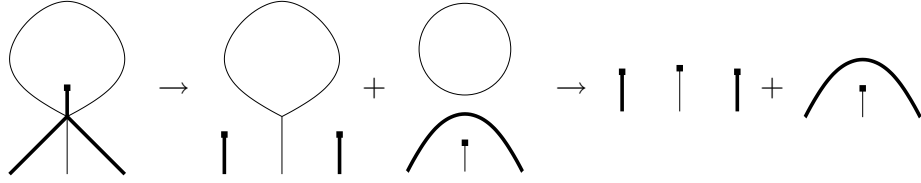


The lines entering the needle around the top are colored with various j all distant from i . The 6-valent picture has second color $i + 1$, but $i - 1$ is also allowed.

We may now reduce the $i + 1$ and $i - 1$ graph of Γ' , since Γ' does not contain the color i . In the first case, the $i \pm 1$ graph has no boundary, so both reduce to the empty graph. In the second case, one of the two has a single boundary line, and the other has none, so one graph reduces to the empty graph, and the other to a single dot. Within the reduced Γ' , this dot may be slid under other lines until it is a boundary dot, as in the proof of the earlier dot move propositions. Thus we may assume that, after reduction, our neighborhood looks like



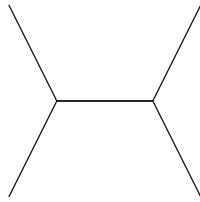
where now Γ' does not contain any of the colors $i - 1, i, i + 1$. Note that Γ' may have arbitrary polynomials in its various regions. But then by Lemma 5.1 we can pull the line forming the needle through all of Γ' , thus completely ignoring Γ' from the picture! There still may be a polynomial in the eye of the needle. We have effectively reduced to the 2-color case on the right, or the one color case on the left. We know the one color case works. To check the 2-color case, we use (3.25) followed by other reduction moves.



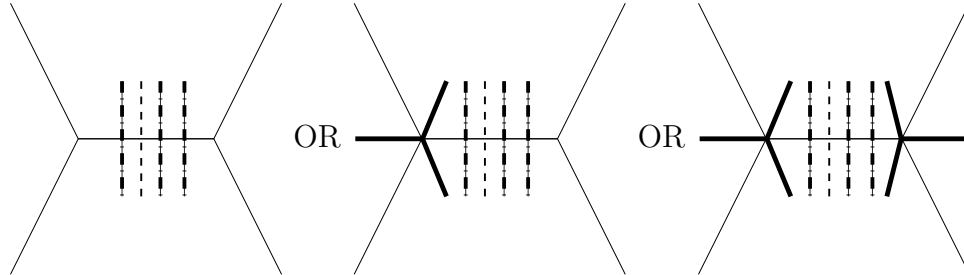
In these final graphs, the i -graph is a dot, as desired. So we may apply the strict i -colored needle move.

Associativity move

We would like to apply associativity to the following subgraph of the i -graph.



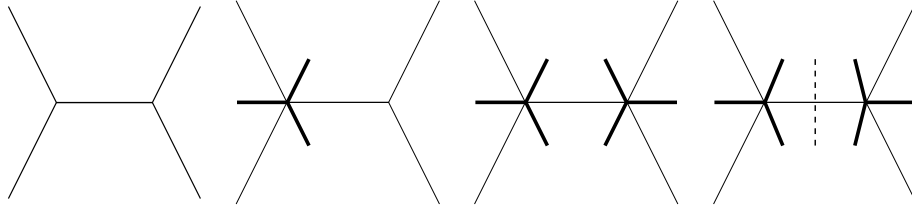
Each trivalent vertex of the i -graph may be either trivalent or 6-valent in Γ . We are forbidding the case when one is 6-valent with $i + 1$ and the other with $i - 1$, so without loss of generality we assume that a 6-valent vertex has $i + 1$ as the other color. The neighborhood of this subgraph in Γ looks like one of the following cases.



Note that all polynomials may be assumed to be outside the neighborhood. Arbitrary lines distant from i may intersect the middle strand. However, we may use Corollary 5.7 to assume that the sequence of lines passing through the middle strand contains at most one instance of $i + 2$ and $i - 2$.

In the first two cases, all such lines may be slid one by one to the right over the trivalent vertex, removing them from the neighborhood. In the third case, all lines not labelled $i + 2$ can be slid to the right or to the left, and since there is at most one line labelled $i + 2$, then at most one line remains. If there had been multiple lines labelled $i + 2$, then additional lines labelled $i + 3$ or higher may have been stuck between them, but we can eliminate this possibility. The case where the 6-valent vertices use $i - 1$ instead is obviously analogous.

So there are really four cases:



In the fourth case, the additional line is $i + 2$. Note that so far, the i -graph has been unchanged.

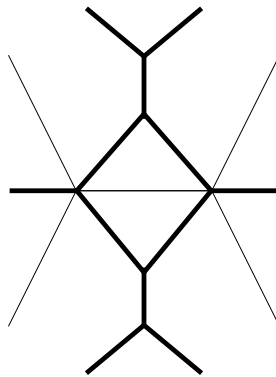
Case I: One color associativity allows the strict i -colored associativity move.

Case II: Double overlap associativity (3.27) allows the strict i -colored associativity move.

The remaining two cases use the same trick: they replace the interior lines on the top and bottom of the graph with the corresponding sum of idempotents, and then resolve each one with double or triple overlap associativity. The remaining two cases will not allow the strict associativity move, only the non-strict one.

Case III: We rewrite equation (3.22), using (3.16), so that there is no polynomial on the bottom.

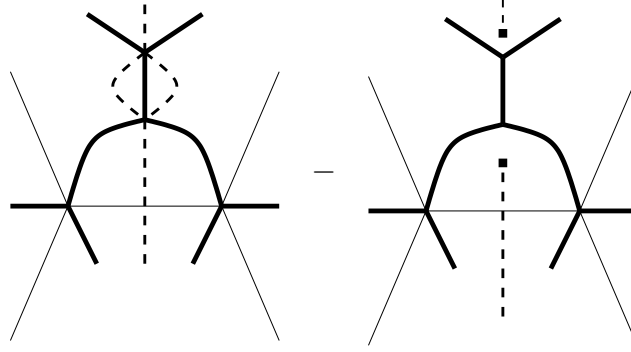
Applying this to the thick edges on top of case III, and symmetrically to the bottom, we get a sum of graphs which either have dots on thick strands connected to 6-valent vertices (implying that the graphs can be simplified) or look like the following, with *no* polynomial in any interior region:



We may apply double overlap associativity to a subgraph of this diagram (ignoring the top and bottom thick trivalent vertices), which will have the final result of applying

i -colored associativity to Γ . The strict move was not applied because of the extra terms which involved dots attached to 6-valent vertices.

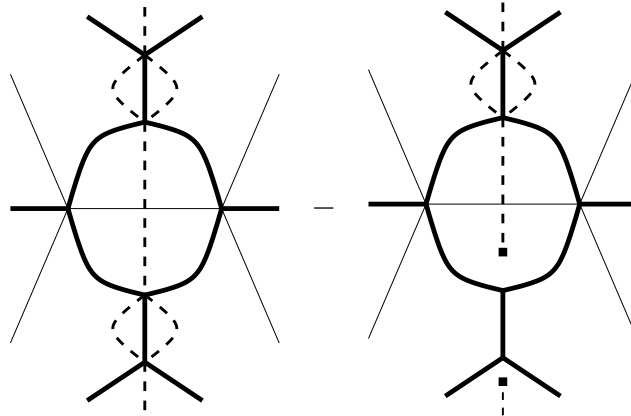
Case IV: Applying equation (3.23) to the $i + 1, i + 2, i + 1$ sequence on top of the graph, we get the difference



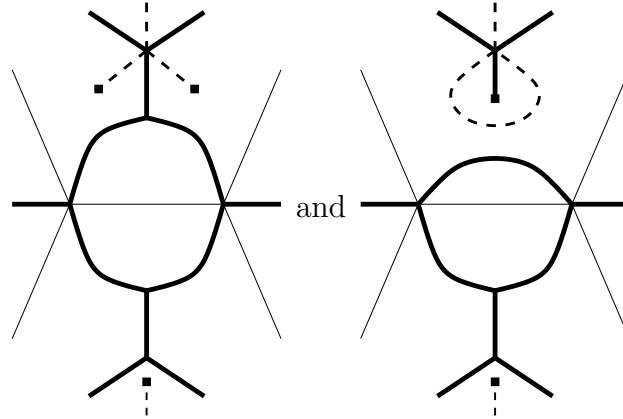
Now we need to show that we can apply associativity to each of these.

For the second picture, we may drag the dot on $i + 2$ through the i -strand, and then a smaller neighborhood of the X looks like case III.

For the first picture, we once again apply (3.23) to the bottom of the graph to get the sum

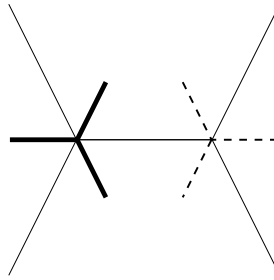


To the subgraph of the left picture which ignores the top and bottom 6-valent vertices, we may apply triple overlap associativity! (Note that this recent operation has performed associativity on the i -graph, while actually increasing the number of total 6-valent vertices, making the $i + 1$ and $i + 2$ graphs more complicated.) In the right picture, we again slide the dot underneath and combine it with the 6-valent vertex on top to get two diagrams



We may apply double overlap associativity to both of these, (3.26) to the first and (3.27) to the second. Again, this results in applying the i -colored associativity move to Γ . \square

Remark 5.10. There does not seem to be a way to simplify the graph



Because of this, our algorithm towards reducing graphs has always been to reduce the smallest color, then reduce the next color within each region delineated by the previous, and so forth, so that at each stage the color $i - 1$ is absent. We could not start with a non-extremal color, because the above diagram might show up.

Despite the pessimism of the previous remark towards reducing non-extremal colors, Proposition 4.15 implies that we can still do a lot. There should be a purely graphical proof of this proposition, but it is complicated by the failure of i -colored associativity when both adjacent colors are present.

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